Child labor, schooling quality and growth*

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Abstract

An overlapping generations model with parental educational choices is built to study the impact of schooling policies and of growth on child labor in a developing country. Households face a trade off between child labor and education in a framework when a minimum of consumption is required.

The model exhibits multiple equilibria. We discuss the existence and the stability of a poverty trap and those of a "child labor" trap, that is characterized by the persistence of child labor over time, even in a growing economy. We then show that poverty plays a prominent part in the rise of child labor, but it is not the major explanation for the existence of a "child labor trap", that occurs with a low quality of schooling or when schooling returns are not big enough.

Key words: education, child labor, poverty trap, growth, schooling, schooling returns

JEL : I20, J24, O11

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1 Introduction

Children work all over the world. According to the ILO, in 2000 at least 190 million of the world's children under age 15 were economically active, with half in Asia (excluding Japan). If we examine the incidence of child labor, Sub-Saharan Africa comes first, with the highest child work ratio (26.4 %). In many developing countries, the number of school years is on average rather low and child labor appears as a mass phenomenon.

TABLE 1. CHILD LABOR:
Regional Estimates of Economically Active
Children ages 5-14 in 2004

	Number of children (in millions)	Work ratio (%)
Asia and the Pacific	122.3	18.8
Latin-America&Caribbean	5.7	5.1
Sub-Saharan Africa	49.3	26.4
Other regions	13.4	5.2
Total	190.7	15.8

Source : ILO (2006).

Moreover, among children going to school in developing countries, up to one-third of the boys and more than two-fifth of the girls are engaged in economic activities on a part-time basis[1] (ILO, 2002).

Child labor is an old and complex problem (see Basu, 1999, for a well documented survey), and appears to be inefficient (see Ranjan, 1999, Balland and Robinson, 2000 for example). What gives rise to child labor and what are its consequence ? What resources may be required to eliminate it ? In the battle against child labor, a set of laws have been tried on national and supranational levels, through international organizations such as the ILO and UNICEF, but the determinants of child labor are rather difficult to evaluate and the choice of an "international labor standard" is complex. For example, enforcing restrictions on child labor through the use of trade sanctions may enhance poverty and do not necessarily solve the problem if it comes from bad economic conditions [2].

[1] These figures probably underestimate child labor, because the ILO statistics do not take into account domestic duties (as cooking, child care, ...).

[2] The question of child labor regulation and of the choice of an international labor standard is rather difficult. Basu (2000) shows for instance that a minimum wage legislation

The purpose of the paper is to focus on some potential child labor determinants in a dynamic framewok. More specifically, two main determinants are considered in the paper : poverty and schooling quality. We provide a theoretical framework, in order to analyse the mechanisms that can give rise to child labor, when there is a trade-off between human capital accumulation and child labor, and we compare the effects of different educational policies on both growth and child labor.

In formal analysis, child labor is generally related to the modeling of household behaviors. Seminar works of Rosenzweig and Evenson (1977) or Parsons and Goldin (1989) tried to explain simultaneously decisions of consumption and child labor, linked to education or fertility, in simple statics framewok, aimed to allow empirical testing. These models of child labor assume that parents take decisions for the whole household, by taking into account for the child and for the effects of total earnings changes on consumption.

However the impact of a trade off between child labor and education on growth is often ignored in the literature, with few exceptions like Glomm (1997) and Baland and Robinson (2000). Baland and Robinson (2000) show that child labor is inefficient, ant that a ban on child labor may be benefit for the society. Actually, when parents used child labor as a substitute for negative bequest (to transfer income from children to parents) or as a substitute for borrowing (to transfer income from the future to present) then child labor is inefficient and hence, a ban on child labor improves the welfare of the society. But these authors don't study the causal links between poverty, schooling and child labor. Glomm (1997) consider two educational regimes, and their respective impact on human capital accumulation, but doesn't consider the effects of poverty in the process.

Nevertheless, empirical studies find evidence that poverty can compel parents to keep children away from school (see for example Jensen and Nielsen, 1997). Psacharopoulos (1997), using data from Latin America, shows that child labor and schooling attainment are substitutes.

The paper contributes to study how poverty and/or a bad schooling quality can generate low-development traps. Different educational policies are discussed in order to show how they may contribute to reduce child labor

in developping countries presents the risk to exacerbating child labor, whereas a chil labor ban may have positive effect on it. Besides, child labor regulation may have positive effects if it reduces fertility, and then the opportunity costs of education (Doepke, 2003, Doepke and Zilibotti, 2003).

and generate growth[3].

A simple OLG model with educational parental choices is built in order to study the effects of poverty on human capital accumulation and on child labor. The dynamics of the model is studied for different schooling systems, in order to characterize the existence, or not, of development traps.

A subsistence consumption is introduced in the model, by means of an intertemporal Stone-Geary utility function. This minimum of consumption represents a poverty line used to identify the part of the population regarded as absolutely poor. As noted by Steger (2000), subsistence consumption denotes a standard of living that allows for the satisfaction of the minimum, physical and mental, basic needs of life. In growth literature, the requirement of subsistence consumption is considered to restrict ability to save. Azariadis (1996) shows that it represents one of the potential causes of poverty traps, and Basu and Van (1998) that it can generate multiple equilibria in a labor market with both child and adult labor.

The model exhibits two determinants for child labor: poverty and the schooling system quality. Two polar educational regimes are considered: a private education regime and a public education regime, in an economy where children are useful as income-earning assets. As noted by the UNESCO (2002) "The goals of expanding education systems and maintaining equitable access to education are inextricably linked to questions of education finance" (p. 12). The level of public and private investment in education varies largely among developing countries, from 1.2 per cent of GDP in Indonesia to 9.9 per cent of GDP in Jamaica or 5.9 in the Philippines, according to the UNESCO (2002).

TABLE 2 :DISTRIBUTION OF EDUCATION
EXPENDITURE BY SOURCES OF FUNDS, 1999

[3] Concrete actions, like the rise of school quality for example (Glewwe, 2002) or building subsidy programs, like the Food-for-Education program (FFE) established successfully in Bangladesh (Ravallion and Woden, 2000), may contribute to fight against poverty and child labor.

% of education expenditures:	Public Sources	Private sources
Chile	55	45
China	56	44
Paraguay	56	44
Indonesia	65	35
Peru	72	28
Argentina	77	23
Jordan	84	16
OECD mean	88	12

Source : OECD/ UNESCO WEI (2002)

The majority of resources in developing countries are focused on primary and secondary education. Private spending often makes a substantial contribution to overall levels of education spending (see Table 2). In this context, it seems interesting to compare both regimes, and to examine their effect in terms of poverty falling and growth.

Different models are studied in the paper, one with an exogenous quality of education, one with a private regime and one with a public regime. The study of the dynamics show that the models exhibit multiple equilibria, and we discuss for each the existence and the stability of a poverty trap (with no growth) and of a "child labor" trap, characterized by the persistence of child labor over time.

It is shown that, for a very poor economy, the public regime can inhibit growth because of the fiscal burden it induces. The quality of the education supply appears as a key element to get out of a "child labor trap". It may require, in a too poor economy, the institution of income transfer programmes.

The rest of the paper is organized as follows. In Section 2 the model is described, under the assumption that parents don't choose the quality of education. In the Section 3, quality of education is introduced and two polar educational regimes are compared: a private education regime and a public one. In Section 4 some concluding remarks are presented.

Proofs are presented in Appendix.

2 Model

A simple model of schooling choices is developped, in which identical individuals live for two periods. There is assumed that parents make decisions for their children, that seems to be realistic for primary and secondary education in developing countries. Their objective is to maximize a utility function that has two arguments: consumption of goods and child cognitive skills (i.e. his or her human capital). For simplicity, it is assumed there is only one child per parent, that lives for two periods. The size of population is normalized to one.

In period t, a child born and may attend school, work or both. If both, the child first goes to school and works after schooling time. When a child works, part or all of the child's earnings is given to his or her parent. Parents have a Stone-Geary utility function, that involves a consumption subsistence term $\overline{c}$. This kind of utility function has be used by Rebelo (1992) in order to explain international differences in rates of growth.

The Stone-Geary utility function is:

$$U(c_t) = \ln(c_t - \overline{c}) + \gamma \ln h_{t+1}, \tag{1}$$

where c_t denotes consumption when old, γ indicates parental tastes for educated children. Parents value educated children for two distincts reasons: educating children can increase parents' consumption and educating children directly affects parents' utility (through γ). Stone-Geary preferences are a simple generalization of COBB-DOUGLAS preferences. In order that the Stone Geary function is defined, we will suppose that $c_t \geq \overline{c} + \eta$, with $\eta > 0$ small.

A simple human capital production function is assumed:

$$h_{t+1} = b\theta_t G(.)h_t^\beta, \tag{2}$$

where $b > 0$ is the "learning efficiency" of the child, $\theta_t \in [\varepsilon, 1]$ the time devoted to school (i.e. the years of schooling), with $\varepsilon > 0$ small (in particular $\varepsilon < (\frac{\overline{c}}{1+\lambda})^{1-\beta}\frac{1}{bQ}$) and h_t the inherited human capital from parents, $\beta \in]0, 1]$. The function G is increasing with school quality, assumed to be exogenous in this section.

Lemma 1 *If there exists t_0 such as $h_{t_0} < \frac{\overline{c}}{1+\lambda}$, then*
(1) $\forall t \geqslant t_0$, $h_t < \frac{\overline{c}}{1+\lambda}$
(2) $\lim_{\varepsilon \to 0}(\lim_{t \to \infty} h_{t0}) = 0$).

Proof. *(1) If $h_{t0} < \frac{\overline{c}}{1+\lambda}$, then $\theta_{t0} = \varepsilon$*
Hence $h_{t0} = \varepsilon bQh_t^\beta < \frac{\overline{c}}{1+\lambda}$ because $\varepsilon < (\frac{\overline{c}}{1+\lambda})^{1-\beta}\frac{1}{bQ}$.
(2) $0 \leqslant h_{t+1} \leqslant \varepsilon bQ(\frac{\overline{c}}{1+\lambda})^\beta \to 0$ when $\varepsilon \to 0$. ■

Lemma 2 *If there exists* t_0 *such as* $h_{t_0} < \frac{\overline{c}}{1+\lambda}$, *then* $\forall t \geqslant t_{0+1}$, $c_t = \overline{c} + \eta$.

Proof. *If there exists* t_0 *such as* $h_{t0} < \frac{\overline{c}}{1+\lambda}$, *then* $\forall t \geq t_{0+1}, c_t = \overline{c} + \eta$.

$c_{t0} = \left[\varepsilon[(1+\lambda)bQh_t^\beta] + (p - \lambda bQh_t^\beta)\right] < \overline{c}$ *for* ε *small enough.*

The constraint $c_t \geq \overline{c} + \eta$ *implies that* $c_{t0} = \overline{c} + \eta$. ■

Family's consumption in t is given by:

$$c_t = y_t - p_t\theta_t + (1 - \theta_t)\lambda h_t, \tag{3}$$

where p_t is the price of schooling, y_t the parental income in period t, $(1-\theta_t)h_t$ the child's incomes when working, and λ is the fraction of that income given to the parent[4]. If $\lambda < 1$, there are frictions on the job market. Another interpretation of this inequality is that children receive, for a same skill level, a lower wage than an adult. In t the family income is: $\Omega_t = y_t + (1 - \theta_t)\lambda h_t$. Parents offer one unit of human capital over the period, then $y_t = h_t$. Educating children has a direct cost for the parents $(p_t\theta_t)$ and an opportunity cost due to the time devoted to school by the child (i.e. $\theta_t\lambda h_t$). The results of the paper go through as long as the child income is a significant part of family income ($\lambda > 0$).

Consider the case where school quality is exogenous, so that θ_t is the only choice variable. From the first order condition, optimal years of schooling are:

$$\theta_t = \frac{\gamma\left[(1+\lambda)h_t - \overline{c}\right]}{(1+\gamma)(p_t + \lambda h_t)}. \tag{4}$$

$\theta_t > 0$ if and only if $(1+\lambda)h_t > \overline{c}$, in other words when child income is not absolutely necessary to get a sufficient income to survive, education is possible. On the contrary, when $(1+\lambda)h_t \leq \overline{c}$, there is "no room for education" and $\theta_t = \varepsilon$.

Moreover if

$$\gamma(h_t - \overline{c}) - \lambda h_t < (1+\gamma)p_t. \tag{5}$$

the optimal schooling time, given by (4), is bounded above by one.

Note that if $\gamma < \lambda$, (5) is always satisfied. In this case, the altruism term is not big enough to avoid child labor, when a part of the child income is useful to the household.

When (5) is not satisfied, $\theta_t = 1$: in this case, parents income are sufficient high, relating to p, $\overline{c}$ and γ, to abolish child labor, and the corner

[4]Equation (3) rules out borrowing and saving. In fact, the study is concerned with primary and secondary education, for which credit markets to finance investments in human capital are seldom available (West, 1991).

solution occurs. These results are conform to empirical facts, that show that the major cause of child labor is poverty, and that an increase in standards of living may reduce child labor (see Edmonds and Turk, 2002, for a study on Vietnam).

When $(1+\lambda)h_t > \overline{c}$, the optimal years of schooling are an increasing function of γ, the parents' taste for educated children, and of parental human capital h_t; θ_t is an decreasing function of $\overline{c}$ and of p_t.

Related to this last result, when $\overline{c} + p_t > h_t$ note that the optimal years of schooling θ_t rise when parents expect to receive a larger proportion (λ) of their children's income from working: it means that a high child contribution to the family income enables part-time labor and not full-time labor; otherwise when $\overline{c} + p_t < h_t$, θ_t is an decreasing function of λ, in this case, child income is not necessary to the family to survive, and it is more required when it becomes more attractive.

2.1 Dynamics

To study the steady-state and the dynamics of the model without educational policy, let's consider that $G(.)$ and p_t are constant over time and respectively equal to Q, an index of educational quality, and p. These assumptions would be relaxed in Section 3. Human capital is the only engine of growth in the model. There are multiple steady-state trajectories, that depend upon the subsistence consumption and of the initial values of the parameters. Three main cases can be distinguished, according to the trade-off between education and child labor made by the parents.

Let $H_1 = \frac{\overline{c}}{(1+\lambda)}$ and $H_2 = \frac{(1+\gamma)p_t + \gamma\overline{c}}{(\gamma-\lambda)}$.

If $H_1 < H_2$, the dynamics of human capital is,

$$\begin{cases} h_{t+1} = b\varepsilon Q h_t^{\beta}, \ H_1 \geq h_t \\ h_{t+1} = \phi_1(h_t), \text{if } H_2 > h_t > H_1 \\ h_{t+1} = \phi_2(h_t), \text{if } H_2 \leq h_t \end{cases}$$

with $\phi_1(h_t) = bQ\frac{\gamma(1+\lambda)h_t - \gamma\overline{c}}{(1+\gamma)(p_t+\lambda h_t)} h_t^{\beta}$ and $\phi_2(h_t) = Qbh_t^{\beta}$.

If not the dynamics is given by

$$\begin{cases} h_{t+1} = bQ\frac{\gamma(1+\lambda)h_t - \gamma\overline{c}}{(1+\gamma)(p_t+\lambda h_t)} h_t^{\beta}, \text{if } h_t > H_1 \\ h_{t+1} = b\varepsilon Q h_t^{\beta}, \ \text{other.} \end{cases}$$

Let us consider that $H_1 < H_2$.

Proposition 3 • *a) If* $bQH_2^{\beta-1} < 1$, *the following situations may occur:*

- *If* $H_2 < x_2$; *there is one unique steady state* h^* *which is semi stable.*
- *If* $H_2 > x_2$; *there are three steady states:* $h^*, H^* \in]H_1, H_2[$ *and* $H^{**} \in]H_1, H_2[$*with* $H^* < H^{**}; h^*$ *is stable,* H^**is unstable and* H^{**}*is stable,*

  where

$$x_2 = \frac{-\left(\begin{array}{c}\lambda\bar{c}\,(2-\beta)+\\(1+\lambda)\beta p\end{array}\right)-\sqrt{\left(\begin{array}{c}\lambda\bar{c}\,(2-\beta)+\\(1+\lambda)\beta p\end{array}\right)^2+4\,(\beta-1)^2\,(1+\lambda)\lambda\bar{c}p}}{2\,(\beta-1)\,(1+\lambda)\lambda}$$

- *b) if* $bQH_2^{\beta-1} > 1$, *there are three steady states:* $h^*, H^* \in]H_1, H_2[$ *and* $H_4 = (Qb)^{\frac{1}{1-\beta}} \in]H_2, \infty[$. h^* *and* H_4 *are stable,* $H^* \in]H_1, H_2[$ *is unstable.*

- *c) Si* $bQH_2^{\beta-1} = 1$, *there are 2 steady states:*h^* *and* H_2. h^* *is stable,* H_2 *is semi-stable.*

Corollary 4 *As* $\varepsilon \to 0, h^* \to 0$.

Corollary 5 ***Poverty trap and no education.*** *When* $bQH_2^{\beta-1} < 1$ *and* $H_2 < x_2$, *the family income is below the poverty line, and there is no room for education. The economy lies in a poverty trap, and no human capital accumulation is possible (ie* $\theta_t = 0$*).*

In this case, parents income are not sufficient enough to pay for education, because they are too close from the poverty line. Child labor is a crucial source of income for poor family ($\theta_t = 0$), and the economy remains in a poverty trap. If there are no, or few, borrowing possiblities for schooling investments, as in most developing countries, a public policy against such a poverty trap would be a distribution of food subsidies[5], like the Food-for-Education Program built in Banglasdesh (Ravallion and Wodon, 2000).

The introduction of a subsistence consumption in the model and of a price for education (p) modifies the standard results of OLG model with education in the logarithmic case. With a low initial stock of human capital (when $H_2 > h_t > H_1$), and for some parameters values, the equilibrium law of motion of h_{t+1} is convex in h_t, and the economy may have both child labor and education on the long run (*ie* $0 < \theta < 1$), as corollary 6 shows it:

[5] This assumption is explored in Section 3.

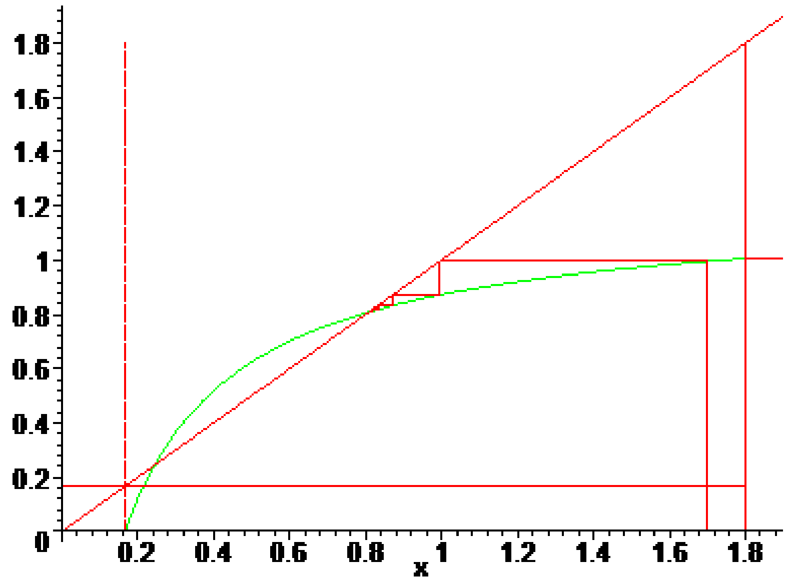

Figure 1: Education and child labor:
with $\beta = 0.01, p = 0.1, b = 1, \bar{c} = 0.3, \lambda = 0.8, \gamma = 1.1, Q = 1.$

Corollary 6 ***Education and child labor.*** *If* $bQH_2^{\beta-1} > 1$ *there exists an unstable steady state where education and child labor can both exist. Convergence to the steady state with only education or only child labor depends on the initial amount of human capital.*

If $bQH_2^{\beta-1} < 1$ *and* $H_2 < x_2$ *there may exists a stable steady state where education and child labor can both exist* (*i.e.* $0 < \theta < 1$).

Figure 1 illustrates the "child labor trap" exhibited in corollary 6.

Corollary 7 ***Education without child labor.*** *When* $bQH_2^{\beta-1} > 1$ *and when initial stock of human capital is high, there is no child labor anymore* ($\theta_t = 1$).

In the case of the Corollary 6, only where the initial stock of human capital is high and when $bQH_2^{\beta-1} > 1$, there is not child labor anymore on the long run ($\theta = 1$), otherwise childlabor will persist permanently ($0 < \theta < 1$).

This situation is little studied in the literature[6], but is frequent in developping countries. For bad initial economic conditions and especially for an inadequate quality of education, the model establishes that it will persist over time.How is it possible then to reduce child labor, to favour schooling and to increase growth ? One solution is to expect a rise of income, in order to be in the case of Corollary 4. Edmonds and Turk (2002) find evidence from Vietnam that an increase of income may rise education participation and to a certain degree diminish child labor. But the model shows that, at times, even if there is an income rise, a poverty trap could emerge. An other solution can be to act on the quality of education, to converge to the third case (where $\theta = 1$). As Glewwe (2002) has pointed it out, one major problem of education in developing countries lies in the bad quality of their schooling system. In next sections, this last point is investigated. The possibility for parents to choose school quality is introduced in order to compare different educational policies and to find some practical solutions to rise schooling attainment.

3 Child labor, education and school quality

School attendance (or conversely child labor) in an low developing country may be explained by poverty, but also by the low quality of schooling that may lead households to substitute work for schooling (see empirical evidence from Zambia in Jensen and Nielsen, 1997). The model can be extended to allow parents to choose school quality[7] in the following way. Like in Glewwe (2002), it is assumed that higher quality implies higher price:

$$p_t = p_0 Q_t, \tag{6}$$

where p_0 is the "base" price of schooling. Q_t may be interpreted as an index of expenditures on quality, and a convenient assumption for a functional form of G is that $G(Q_t) = Q_t^{\alpha}$, with $\alpha < 1$ (see Glewwe 2002 for a use of this specification for example). Two polar educational regimes are successively considered: a private education regime, where all schooling decisions are

[6] Perotti (1993) for example find that $\theta = 0$ or $\theta = 1$.The intermediate case ($0 < \theta < 1$) is not considered, whereas the ILO statistics show that it is a general case in developping countries.

[7] As noted by Glewwe (2002) "School quality is likely to be endogenous. Even in rural areas of low-income countries, where villages often have only one school and are too far apart from children to attend school in a neighbouring village, parents may be able to influence school quality of the sole local school through the parent-teacher association", p. 443.

made individually by the parent, i.e. each parent chooses the child's school time and the quality of school; a public education regime, where parents only choose their child's school time, and education quality depends on taxes.

3.1 Human capital accumulation under private regime

In the private education regime all schooling decisions are made individually by parents, i.e. parents choose the child's school time and the quality of the schools. The family budget constraint at time t is:

$$c_t + p_0 Q_t = \Omega_t$$

The problem of parents becomes:

$$\begin{cases} \underset{c_t,\theta_t,Q_t}{Max} \ln(c_t - \overline{c}) + \gamma \ln h_{t+1} \\ \text{s.t.} \begin{cases} h_{t+1} = b\theta_t Q_t^{\alpha} h_t^{\beta}, \\ c_t = h_t - p_0 Q_t \theta_t + (1-\theta_t)\lambda h_t. \\ \varepsilon \leq \theta_t \leq 1 \\ c_t \geq \overline{c} + \eta \end{cases} \end{cases},$$

Lemma 8 *If $\frac{\overline{c}}{1+\lambda} < h_t < \frac{(1-\alpha)\gamma}{\gamma[1-\alpha(1+\lambda)]-\lambda}\overline{c}$, from the first order condition, optimal school quality and optimal years of schooling are respectively:*

$$\begin{aligned} Q_t &= \frac{\alpha\lambda h_t}{p_0(1-\alpha)} \\ \theta_t &= \frac{\gamma(1-\alpha)\left[(1+\lambda)h_t - \overline{c}\right]}{(\gamma+1)\lambda h_t}, \end{aligned}$$

If $\frac{\overline{c}}{1+\lambda} > h_t$, optima satisfy

$$\begin{aligned} Q_t &= \frac{\alpha\lambda h_t}{p_0(1-\alpha)} \\ \theta_t &= \varepsilon \end{aligned}$$

If $h_t > \frac{(1-\alpha)\gamma}{\gamma[1-\alpha(1+\lambda)]-\lambda}\overline{c}$, optima satisfy

$$\begin{aligned} Q_t &= \left[\frac{\gamma\alpha}{p_0(1+\alpha\gamma)}\right](h_t - \overline{c}), \ \textit{if} \ \frac{(1-\alpha)\gamma}{\gamma\left[1-\alpha(1+\lambda)\right]-\lambda} > 1 \\ \theta_t &= 1, \end{aligned}$$

Notice that condition $\frac{(1-\alpha)\gamma}{\gamma[1-\alpha(1+\lambda)]-\lambda} > 1$ is always satisfied.

PROOF: The First Order Conditions give

$$\begin{aligned}\theta_t &= \frac{\gamma(1-\alpha)\left[(1+\lambda)h_t-\overline{c}\right]}{(\gamma+1)\,\lambda h_t}, \\ Q_t &= \frac{\alpha\lambda h_t}{p_0(1-\alpha)},\end{aligned} \tag{7}$$

θ_t must be bounded by one, hence (7) applies when,

$$\frac{\overline{c}}{1+\lambda} < h_t < \frac{(1-\alpha)\,\gamma}{\gamma\left[1-\alpha(1+\lambda)\right]-\lambda}\overline{c}. \tag{8}$$

If $h_t \geqslant \frac{(1-\alpha)\gamma}{\gamma[1-\alpha(1+\lambda)]-\lambda}\overline{c}$, it is optimal for parents to choose the maximal amount of schooling (i.e. the corner solution occurs). This "threshold" increases with $\overline{c}$, and λ. Note that θ_t is a decreasing function of α and an increasing function of h_t and of λ.

It should be noticed that the condition to have $\theta_t > 0$ if $(1+\lambda)h_t > \overline{c}$ is the same than in the previous case.

Let $H_1 = \frac{\overline{c}}{1+\lambda}$ and $\widehat{H}_2 = \frac{(1-\alpha)\gamma}{\gamma[1-\alpha(1+\lambda)]-\lambda}\overline{c}$.

Hypothesis 1. *It is assumed that* $\gamma\left(1-\alpha\left(1+\lambda\right)\right)-\lambda > 0$.
This assumption is necessary to have $\widehat{H}_2 > 0$.

The quality of schooling desired by parents is an increasing function of their own human capital h_t and of λ. It decreases with p_0, the "base" price of schooling, in other words, if the price of education is rather high, parents don't want to pay more to improve quality of education.

The law of motion of human capital is

$$\begin{cases} h_{t+1} = b\varepsilon\frac{\alpha\lambda}{p_0(1-\alpha)}h_t^{\alpha+\beta}, \text{ if } H_1 > h_t \\ h_{t+1} = \psi_1(h_t), \text{ if } H_1 < h_t < \widehat{H}_2 \\ h_{t+1} = \psi_2(h_t), \text{ if } h_t > \widehat{H}_2. \end{cases}$$

where $\psi_1(h_t) = b\frac{\gamma(1-\alpha)}{(\gamma+1)\lambda}\left[\frac{\alpha\lambda}{p_0(1-\alpha)}\right]^{\alpha}\left[(1+\lambda)h_t-\overline{c}\right]h_t^{\alpha+\beta-1}$

and $\psi_2(h_t) = b\left[\frac{\gamma\alpha}{p_0(1+\alpha\gamma)}\right]^{\alpha}(h_t-\overline{c})^{\alpha}h_t^{\beta}$; $\psi_2(h_t)$ is defined on $[\widehat{H}_2,\infty[$. Moreover, we can notice that $\psi_1(\widehat{H}_2) = \psi_2(\widehat{H}_2)$.

Proposition 9 *1. In the case where there are decreasing returns in the human capital accumulation function* $(\alpha+\beta-1<0)$,

$\widehat{h}$ *is a steady state of the dynamics (poverty trap) and it is stable. It may exist other steady states:*

- *If* $\widehat{H}_2 < x_2^*$,
  - *If* $\psi_1(\widehat{H}_2) < \widehat{H}_2$ *then there is no steady state in* $]H_1, \widehat{H}_2[$ *and*
    - *if* $g(x_2^*) > 1/G$ *there are two steady states* $H_3^* < H_4^*$ *in* $]\widehat{H}_2, \infty[$ *;*
      H_3^* *is unstable;* H_4^* *is stable*
    - *if* $g(x_2^*) > 1/G$ *there is no solution in* $]\widehat{H}_2, \infty[$
  - *If* $\psi_1(\widehat{H}_2) > \widehat{H}_2$ *then there is one steady state* H_1^* *in* $]H_1, \widehat{H}_2[$ *and one steady state* H_2^* *in* $]\widehat{H}_2, \infty[$
    H_1^* *is unstable,* H_2^* *is stable*
- *If* $x_2^* < \widehat{H}_2 < x_1^*$
  - *If* $\phi_1\left(\widehat{H}_2\right) < \widehat{H}_2$ *there are no steady state*
  - *If* $\phi_1\left(\widehat{H}_2\right) > \widehat{H}_2$ *there are is one steady state* H_1^* *in* $]H_1, \widehat{H}_2[$ *and one steady state* H_2^* *in* $]\widehat{H}_2, \infty[$
    H_1^* *is unstable,* H_2^* *is stable*
- *If* $x_1^* < \widehat{H}_2$,
  - *If* $f(x_1^*) < 1/F$ *then there is no steady state in* $]H_1, \widehat{H}_2[$ *and there is no steady states in* $]H_2, \infty[$ *;*
  - *If* $f(x_1^*) > 1/F$ *and* $\psi_1(\widehat{H}_2) < \widehat{H}_2$ *there are two steady states* $H_1^* < H_2^*$ *in* $]H_1, \widehat{H}_2[$ *and there is no solution in* $]\widehat{H}_2, \infty[$
    - H_1^* *is unstable,* H_2^* *is stable*
  - *If* $f(x_1^*) > 1/F$ *and* $\psi_1(\widehat{H}_2) > \widehat{H}_2$ *there is one steady state* H_1^* *in* $]H_1, \widehat{H}_2[$ *and one steady state* H_2^* *in* $]\widehat{H}_2, \infty[$
    H_1^* *is unstable,* H_2^* *is stable*

2 In the case where the are increasing returns in the human capital accumulation function $(\alpha+\beta-1>0)$,

h^* *is a steady state of the dynamics (poverty trap) and itis unstable. It may exists other steady states :*

- *If* $\psi_1\left(\widehat{H}_2\right) > \widehat{H}_2$ *there is a unique steady state* H_1^* *in* $]H_1, \widehat{H}_2[$, *which is unstable*
- *If* $\psi_1\left(\widehat{H}_2\right) < \widehat{H}_2$ *there is a unique steady state* H_2^* *in* $]\widehat{H}_2, \infty[$, *which is unstable*

Corollary 10 *When* $\varepsilon \to 0$, $h^* \to 0$.

Corollary 11 ***Child labor and education.*** *When there are decreasing returns in the human capital accumulation* ($\alpha+\beta-1<0$) *there may exist a steady state with education and child labor that is stable. In the case where* $\alpha+\beta-1>0$, *the steady state with both education and child labor, when it exists, is unstable.*

The mechanism describes in the paper contributes to show that a private education regime may create good incentives for schooling and may help to reduce child labor, except if the economy lies in a poverty trap. In this case, other educational policies are necessary.

3.2 Human capital accumulation under public education

In the public education regime, the government is assumed to collect income taxes at a uniform rate (τ_t) from the labor income of the old. Income of the young is supposed to exempt from taxation. Tax revenues are used to finance education and all children have access to the same quality of education at price p_0, such as $p_0 Q_t$.

Under these assumptions the family budget constraint becomes:

$$c_t = (1-\tau_t)y_t - p_0 Q_t \theta_t + (1-\theta_t)\lambda h_t.$$

The quality of education to which all children have access becomes: $Q_t = \tau_t Y_t$, where Y_t denotes the aggregate (average) variables. Since the size of the population is normalized to one, it comes that $Y_t = h_t$. Under public education each individual, when old, solves the following maximization problem:

$$\underset{\theta_t, c_t}{Max} \ln(c_t - \overline{c}) + \gamma \ln h_{t+1},$$

s.t.

$$c_t = (1-\tau_t)y_t - p_0 \tau_t Y_t \theta_t + (1-\theta_t)\lambda h_t,$$

$$h_{t+1} = b\theta_t \tau_t^{\alpha} h_t^{\alpha+\beta},$$

$$\eta + \overline{c} \leq c_t; \varepsilon \leq \theta_t \leq 1.$$

Parents, reasonably, consider the tax rate τ_t as given. Consequently, the only choice variable is θ_t.

From the F.O.C.,

$$\theta_t = \frac{\gamma\left[(1-\tau_t+\lambda)h_t - \overline{c}\right]}{(1+\gamma)\left[p_0\tau_t + \lambda\right]h_t}$$

Hypothesis 2. *The proportional tax is inferior to* $1+\lambda$.

This assumption is necessary to avoid a fiscal burden that would inhibit schooling and growth.

Discussion:

$\theta_t > 0$ if and only if $h_t > \frac{\overline{c}}{1-\tau_t+\lambda}$. This opens the possibility to make subsidies and not a fiscal policy (when $\tau_t < 0$).

$\theta_t < 1$ when $h_t < \frac{\gamma\overline{c}}{\gamma(1-\tau_t)-\lambda-(1+\gamma)p_0\tau_t}$. Note that the human capital threshold to avoid child labor rises with $\overline{c}, p_0$ and with the tax rate τ_t.

The optimal years of schooling θ_t are an increasing function of the parental human capital h_t. Besides, they fall with the tax rate if $h_t > \frac{p_0\overline{c}}{\lambda+p_0+P_0\lambda}$, and rise with τ when $h_t < \frac{p_0\overline{c}}{\lambda+p_0+P_0\lambda}$. In this last case, it would be possible to enhance growth with the tax, through its positive impact on education quality and the incentive motives it generates.

There is a multiplicity of steady state trajectories, according to the initial value of h_t. To study them, let's assume that the income tax rate is constant on the long run. Let $\overline{H}_1 = \frac{\overline{c}}{1-\tau_t+\lambda}$ and $\overline{H}_2 = \frac{\gamma\overline{c}}{\gamma(1-\tau)-\lambda-(1+\gamma)p_0\tau}$. The law of motion of human capital is

$$\begin{cases} h_{t+1} = b\varepsilon\tau_t^{\alpha}h_t^{\alpha+\beta}, \text{ if } \overline{H}_1 > h_t \\ h_{t+1} = \varkappa_1(h_t), \text{ if } \overline{H}_1 < h_t < \overline{H}_2 \\ h_{t+1} = \varkappa_2(h_t), \text{ if } h_t > \overline{H}_2. \end{cases}$$

with $\varkappa_1(h_t) = b\tau^{\alpha}\left[\frac{\gamma[(1-\tau+\lambda)h_t-\overline{c}]}{(1+\gamma)[p_0\tau+\lambda]}\right]h_t^{\alpha+\beta-1}$ and $\varkappa_2(h_t) = b\tau_t^{\alpha}h_t^{\alpha+\beta}$.

Proposition 12 *h^* is a stable steady state of the dynamics (poverty trap), it may exist other steady states:*

- *Let us assume that* $(\alpha+\beta-1) < 0$,

  h^* is a stable steady state of the dynamics (poverty trap), it may exist other steady states:

- *If* $\frac{(\alpha+\beta-2)\overline{c}}{(\alpha+\beta-1)(1-\tau+\lambda)} > \overline{H}_2$
  - *If* $\chi_1(\overline{H}_2) < \overline{H}_2$, $\overline{H}_1$ *is the only steady state of the dynamics; it is stable.*
  - *If* $\chi_1(\overline{H}_2) > \overline{H}_2$, *the steady states of the dynamics are* $\overline{H}_1$, $H^* \in]\overline{H}_1, \overline{H}_2[$ *and* $H_2^* \in]\overline{H}_2, \infty[$;

    $\overline{H}_1$ *is stable,* $\overline{H}^*$ *is unstable and* $\overline{H}_2^*$ *is stable*
- *If* $\frac{(\alpha+\beta-2)\overline{c}}{(\alpha+\beta-1)(1-\tau+\lambda)} < \overline{H}_2$
  - *If* $\varkappa_1(\overline{H}_2) > \overline{H}_2$, *the steady states of the dynamics are* $\overline{H}_1$, $H^* \in]\overline{H}_1, \overline{H}_2[$ *and* $H_2^* \in]\overline{H}_2, \infty[$;

    $\overline{H}_1$ *is stable,* $\overline{H}^*$ *is unstable and* $\overline{H}_2^*$ *is stable*
  - *If* $\varkappa_1(\overline{H}_2) < \overline{H}_2$, *and* $f(\frac{(\alpha+\beta-2)\overline{c}}{(\alpha+\beta-1)(1-\tau+\lambda)}) < F$, $\overline{H}_1$ *is the only steady state of the dynamics; it is stable.*
  - *If* $\chi_1(\overline{H}_2) < \overline{H}_2$, *and* $f(\frac{(\alpha+\beta-2)\overline{c}}{(\alpha+\beta-1)(1-\tau+\lambda)}) > F$,*the steady states of the dynamics are* $\overline{H}_1$, $H^* \in]\overline{H}_1, \overline{H}_2[$ *and* $H_1^* \in]\overline{H}_1, \overline{H}_2[$ *;*

    $\overline{H}_1$ *is stable,* H^* *is unstable and* H_1^* *is stable.*

- *Let us assume that* $(\alpha + \beta - 1) > 0$,

  h^* *is a unstable steady state of the dynamics (poverty trap), it may exist other steady states:*

  - *If* $\chi_1(\overline{H}_2) > \overline{H}_2$, *the steady states of the dynamics are* $\overline{H}_1$ *and* $H^* \in]\overline{H}_1, \overline{H}_2[$.

    $\overline{H}_1$ *is stable,* $\overline{H}^*$ *is unstable.*
  - *If* $\chi_1(\overline{H}_2) < \overline{H}_2$, $\overline{H}_1$ *is the only steady state of the dynamics; it is stable.*

Corollary 13 *If* $\varepsilon \to 0, h^* \to 0$.

Corollary 14 *With a public education regime, when the initial human capital level is too low, the economy converges to a poverty trap (with no education). In this particular case, the fiscal burden will contribute to inhibit* *growth. Otherwise, if the economy avoid the poverty trap, child labor may totally disappear on the long run.*

For some value of the parameters, a stable steady state with both education and child labor may exist.

For $h_t < \overline{H}_1$, the economy is too poor and lies in a poverty trap. It should be noticed that the poverty trap occurs more easely in this case than with the private regime. This is due to the tax, that diminishes family's income.

The dynamics in the case where $\overline{H}_1 < h_t < \overline{H}_2$ is quite similar to the second case of the dynamics of the private education system (when $H_1 < h_t < \widehat{H}_2$). Here once again, the economy converges more easely to a poverty trap, because of the fiscal burden ($\overline{H}_1 > H_1$). In comparison with the case with exogenous quality of education, it is possible with a public regime to act on the tax to put an end to child labor. Nevertheless, the effect of the income tax is ambiguous, because of its impact on the trade-off between education and child labor: for $h_t < \frac{p_0 \overline{c}}{\lambda + p_0 + p_0 \lambda}$, it appears that the optimal years of education are an increasing function of the tax rate, and this policy contributes to enhance growth (the *quality* effect dominates the *income* effect of the tax). Otherwise, a too high level of tax may inhibit growth (the *income* effect of the tax dominates the *quality* effect), and may generate a poverty trap.

In the public regime, when $h_t > \overline{H}_2$, there is no child labor anymore and $\theta = 1$. In this case, if $\alpha + \beta < 1$, there is one unique steady state which is stable and which rises with the tax level. If $\alpha + \beta = 1$, there is a standard endogenous growth regime, and the growth rate increases with the tax level. The major difference with the private regime is that a high initial stock of human capital generates a potentially higher growth with a public education regime, that does not rely on $\overline{c}$. Consequently, no cycles may occur with a public regime and it is possible to favour growth by the way of the tax.

Different educational policies may be implented in order to favour education and, indirectly, reduce child labor. For a poor economy above the poverty line ($\overline{H}_1 < h_t < \overline{H}_2$), the private regime appears to be more efficient to fight against child labor, because the poverty trap occurs less easely. This is coherent with empirical facts in developing countries, where private education systems are rather developped, and are often financed at communauty levels. In this case, the public system may create not enough incentives ;

moreover when households are very poor, they can not bear a too heavy fiscal burden and only a redistributive policy could be efficient. Nevertheless, if the economy is not too poor and can finance a public system by taxes, this one may be more efficient that the private system to contribute to reduce child labor on the long run.

More generally, if the economy lies in a low-development trap, a transitory public aid (subsidies, ...) would be necessary to help the economy to get out the poverty trap. In this case, the institution of income transfer programmes to defray the cost to households of transferring children from school can be an issue (like the Bosla Escola programme in Brazil for instance).

4 Concluding remarks

The model contributes to highlight the link between poverty, education and child labor. Under bad economic conditions, it appears that parents may decide to keep children away from school. In such a context, trade sanctions or repressive laws seem not to be the right solution to fight against child labor. Government policies have to act mainly on growth and on poverty.

Different education policies are considered in the paper and a subsistence consumption is introduced to define a poverty line.

When the quality of education is too low or when the households income are below the poverty line, a low-development trap occurs. An improvment of the quality of education through a private education regime or a public education regime may enhance human capital growth and reduce child labor.

Nevertheless, if the stock of human capital is low and above the poverty line, the private regime may be more efficient that the public education regime through the incentives it creates, linked to the quality of education. Besides in this case, a public aid can be necessary to finance a public education system, in order to avoid a too heavy fiscal burden that would induce a poverty trap. But if the economy is not too poor, the public regime may be as good as the private system one, or better if the educational quality is high enough.

These results are coherent with empirical facts and may give explanations to the relatively important development of private schooling systems in developing countries at communauty or local levels.

It is also shown that when the economy is below the poverty line, none of the education regimes considered in the paper enables to get out the low-development trap. In this case only subsidies policy would contribute to enhance growth.

Cross-section data from rural India will be used to study the sensitivity of child labor to the household income, and to study the incentives generated by the schooling system. These empirical results will be presented in the final draft of the paper.

It would be also interesting to introduce savings in the model in order to study how a social protection system may contribute to reduce child labor. This would be done in further work.

References

[1] ASHAGRIE K., 1993, Statistics on Child labor, *Bulletin of labor Statistics*, 3, International Labor Organization, Geneva.

[2] AZARIADIS C., 1996, The Economics of Poverty Traps: Part One: Complete Markets, *Journal of Economic Growth*, 1(4), 449-496.

[3] BALAND J.-M. and J.A. ROBINSON, 2000, Is Child Labor Inefficient?, *Journal of Political Economy*, 108(4), 663-679.

[4] BASU A.M. and P.H. VAN, 1998, The Economics of Child Labor, *American Economic Review*, 88(3), 412-27.

[5] BEN-DAVID Dan, 1998, Convergence Clubs and Subsistence Economies, *Journal of Development Economics*, 55, 155-171.

[6] BOMMIER A. and P. DUBOIS, 2004, Rotten Parents and Child Labor, *Journal of Political Economy*, 112 (1), 240-248.

[7] BOURGUIGNON F. and P.-A. CHIAPPORI, 1994, The Collective Approach to Household Behavior, in *Measurement of Household Welfare*, R. Blundell, I. Preston, and I. Walker Eds., Cambridge University Press, UK.

[8] DASGUPTA P., 1998, The Economics of Poverty in Poor Countries, *Scandinavian Journal of Economics*, 100(1), 41-68.

[9] DOEPKE M., 2003, Accounting for Fertility during the Transition to Growth, mimeo UCLA.

[10] DOEPKE M. and F. ZILIBOTTI, 2003, The Macroeconomics of Child Labor Regulation, mimeo UCLA.

[11] EDMONDS E. and C. TURK, 2002, Child Labor in Transition in Vietnam, *World Bank Policy Research Paper* n°2774, forthcoming in *Economic Growth, Poverty and Household Welfare: Policy Lessons from Vietnam*, P. Glewwe, N. Agrawal and D. Dollar Eds., World Bank, Washington DC.

[12] EHRLICH I. and F.T. LUI, 1991, Intergenerational Trade, Longevity and Economic Growth, *Journal of Political Economy*, 99, 1029-1059.

[13] FYFE A., 1989, *Child Labor*, Polity Press, Cambridge.

[14] GLEWWE P., 2002, School and Skills in Developing Countries: Education Policies and Socioeconomic Outcomes, *Journal of Economic Literature*, XL, 436-482.

[15] GLOMM G., 1997, Parental Choice of Human Capital Investment, *Journal of Development Economics*, 53, 99-114.

[16] GLOMM G. and B. RAVIKUMAR, 1992, Public versus Private Investment in Human Capital: Endogenous Growth and Income Inequality, *Journal of Political Economy*, 100, 818-834.

[17] International Labour Organization, 1996, *Child Labour Surveys,* International Programme on the Elimination of Child Labour (IPEC), ILO, Geneva.

[18] International Labour Organization, 2006, *Global Child Labour Trend,* International Programme on the Elimination of Child Labour (IPEC), ILO, Geneva.

[19] JENSEN P. and H. S. NIELSEN, 1997, Child Labour or School Attendance? Evidence from Zambia, *Journal of Population Economics*, 10, 407-424.

[20] PARSONS D.O. and C. GOLDIN, 1989, Parental Altruism and Self-Interest: Child Labor Among Late Nineteenth-Century American Families, *Economic Inquiry*, 27(4), 637-59.

[21] PSACHAROPOULOS G., Child Labor versus Educational Attainment. Some Evidence from Latin America, 1997, *Journal of Population Economics*, 10, 377-386.

[22] PSACHAROPOULOS G., 1989, Time Trends of the Return to Education: Cross-National Evidence, *Economics of Education Review*, 8, 225-231.

[23] RAVALLION M. and Q. WODON, 2000, Does Child Labour Displace Schooling? Evidence on Behavioural Responses to an Enrollment Subsidy, *The Economic Journal*, 110(462), 158-175.

[24] RAY R., 2000, Analysis of Child Labour in Peru and Pakistan: A Comparative Study, *Journal of Population Economics*, 13, 3-19.

[25] REBELO S., 1992, Growth in Open Economies, *Carnegie-Rochester Conference Series on Public Policy,* 36, 5-46.

[26] ROSENZWEIG M.R. and R. EVENSON, 1977, Fertility, Schooling and the Economic Contribution of Children in Rural India: An Economic Analysis, *Econometrica*, 45(5), 1065-79.

[27] STEGER T.M., 2000, Economic Growth with Subsistence Consumption, *Journal of Development Economic*s, 62, 343-361.

[28] UNESCO, 2002, *Financing Education-Investments and Returns, Analysis of the World Education Indicators,* Unesco Institute for Statistics and OECD Report.

[29] WEST E., 1991, Public Schools and Excess Burdens, *Economics of Education Review*, 10, 159-169.

APPENDIX

PROOF PROPOSITION 1

1. Steady states:

   (a) Steady states in $]H_2, \infty[$ are solutions of $\phi_2(h) = h$, that is $h = (Qb)^{\frac{1}{1-\beta}}$ .Notice that $(Qb)^{\frac{1}{1-\beta}} > H_2$ if and only if $bQH_2^{\beta-1} > 1$
   If $(Qb)^{\frac{1}{1-\beta}} < H_2$, the dynamics has no steady state in $]H_2, \infty[$;
   if $(Qb)^{\frac{1}{1-\beta}} > H_2$, the dynamics has a unique steady state $(Qb)^{\frac{1}{1-\beta}} \in]H_2, \infty[$;
   if $(Qb)^{\frac{1}{1-\beta}} = H_2$, H_2 is the unique steady state in $[H_2, \infty[$.

   (b) Steady states in $]H_1, H_2[$ are solutions of

   $$bQ\frac{\gamma}{\gamma+1}f(x) = 1,$$

   i. where $f(x) = \frac{(1+\lambda)x-\bar{c}}{(p+\lambda x)}x^{\beta-1}$.

$$f'(x) = \frac{x^{\beta-2}}{(p+\lambda x)}\left(\frac{(1+\lambda)p+\lambda\bar{c}}{(p+\lambda x)}x + ((1+\lambda)x-\bar{c})(\beta-1)\right)$$

The sign of $f'(x)$ is the same as the sign of
$P(x) = (\beta-1)(1+\lambda)\lambda x^2 + x\begin{pmatrix}\lambda\bar{c}(2-\beta)+\\(1+\lambda)\beta p\end{pmatrix} - \bar{c}(\beta-1)p$.
Let $\Delta = \begin{pmatrix}\lambda\bar{c}(2-\beta)+\\(1+\lambda)\beta p\end{pmatrix}^2 + 4(\beta-1)^2(1+\lambda)\lambda\bar{c}p > 0$
and x_2 be the positive root of $P(x)$;

$$x_2 = \frac{\begin{pmatrix}\lambda\bar{c}(2-\beta)+\\(1+\lambda)\beta p\end{pmatrix} + \sqrt{\begin{pmatrix}\lambda\bar{c}(2-\beta)+\\(1+\lambda)\beta p\end{pmatrix}^2 + 4(\beta-1)^2(1+\lambda)\lambda\bar{c}p}}{2(1-\beta)(1+\lambda)\lambda}$$

The graph of $f(x)$ is the following:

	H_1		x_2	∞
$bQ\frac{\gamma}{\gamma+1}f'(x)$		$+$		$-$
$bQ\frac{\gamma}{\gamma+1}f(x)$	0	$\nearrow$	$bQ\frac{\gamma}{\gamma+1}f(x_2)$	$\searrow 0$

Let us distinguish the following cases:

- If $H_2 < x_2$
  we then have to distinguish the following cases:
  - if $bQ\frac{\gamma}{\gamma+1}H_2^{\beta-1} < 1$; there are no steady states in $]H_1, H_2[$
  - Soit $bQ\frac{\gamma}{\gamma+1}H_2^{\beta-1} > 1$;there is a unique steady state H^* in $]H_1, H_2[$.
- If $H_2 > x_2$
  we then have to distinguish the following cases:
  - If $f(x_2) < 1$;there are no steady states in $]H_1, H_2[$
  - If $f(x_2) > 1$;
    we then have to distinguish the following cases:
    * If $bQ\frac{\gamma}{\gamma+1}H_2^{\beta-1} > 1$;there is a unique steady state H^* in $]H_1, H_2[$.
    * Soit $bQ\frac{\gamma}{\gamma+1}H_2^{\beta-1} < 1$;there are two steady states $H^*$and $H^{**}$with $H^* < H^{**}$in $]H_1, H_2[$.

2. Dynamics: To study the dynamics, just notice that $\phi_1(x)$, and $\phi_2(x)$ are increasing function of x; and that $\phi_1(H_1) = 0$

PROOF PROPOSITION 2

1. Steady states in $]H_1, \widehat{H}_2[$ are solutions of $f(x) = \frac{1}{F}$ where $F = b\frac{\gamma(1-\alpha)}{(\gamma+1)\lambda}\left[\frac{\alpha\lambda}{p_0(1-\alpha)}\right]^{\alpha}$ and $f(x) = [(1+\lambda)x - \overline{c}]\, x^{\alpha+\beta-2}$.

$$f^{'}(x) = x^{\alpha+\beta-3}\left((\alpha+\beta-1)\, x(1+\lambda) - (\alpha+\beta-2)\,\overline{c}\right)$$

Let $x_1^* = \frac{(\alpha+\beta-2)\overline{c}}{(\alpha+\beta-1)(1+\lambda)}$ be the solution of $f^{'}(x^*) = 0$;

(a) If $(\alpha+\beta-1) > 0$, $x_1^* < H_1$.The graph of $f(x)$ is the following:

	H_1		∞
$f^{'}(x)$			$+$
$f(x)$	0	$\nearrow$	∞

i. if $f(\widehat{H}_2) > \frac{1}{F}$,there exists a unique solution H^* in $]H_1, \widehat{H}_2[$ such that $f(H^*) = \frac{1}{F}$

ii. otherwise there are no steady states in $]H_1, \widehat{H}_2[$.

(b) If $(\alpha+\beta-1)<0$, $x_1^*>H_1$. The graph of $f(x)$ is the following:

$$\begin{array}{ccccc} & H_1 & & x_1^* & \infty \\ f'(x) & & + & 0 & - \\ f(x) & {}^0 & \nearrow & f(x^*)>0 & \searrow 0 \end{array}$$

i. If $\widehat{H}_2 < x_1^*$;

A. If $f(\widehat{H}_2) < 1/F$ there is no solution in $]H_1, \widehat{H}_2[$;

B. If $f(\widehat{H}_2) > 1/F$ there is one solution H^* in $]H_1, \widehat{H}_2[$.

ii. If $\widehat{H}_2 > x_1^*$;

A. if $f(x_1^*) < 1/F$ there is no solution in $]H_1, \widehat{H}_2[$.

B. if $f(x_1^*) > 1/F$ and if $f(\widehat{H}_2) < 1/F$ there are two solution H_3 and H_4 in $]H_1, \widehat{H}_2[$

C. if $f(x_1^*) > 1/F$ and if $f(\widehat{H}_2) > 1/F$ there is one solution H_3 in $]H_1, \widehat{H}_2[$.

2. Steady states in $]\widehat{H}_2, \infty[$ are solutions of $g(x) = \frac{1}{G}$ where $G = b\left[\frac{\gamma\alpha}{p_0(1+\alpha\gamma)}\right]^{\alpha}$ and $g(x) = (x-\overline{c})^{\alpha} x^{\beta-1}$.

$$g'(x) = x^{\beta-2}(x-\overline{c})^{\alpha-1}\left((\alpha+\beta-1)\,x - (\beta-1)\,\overline{c}\right)$$

Let $x_2^* = \frac{(\beta-1)\overline{c}}{(\alpha+\beta-1)}$ the solution of $g\prime(x) = 0$

(a) If $(\alpha+\beta-1)<0$ the graph of $g(x)$ is the following:

$$\begin{array}{ccccc} & 0 & & x_2^* & \infty \\ g'(x) & & + & & - \\ g(x) & {}^0 & \nearrow & g(x_2^*)>0 & \searrow 0 \end{array}$$

i. If $\widehat{H}_2 < x_2^*$

A. If $g(\widehat{H}_2) > 1/G$,there is one solution in $]\widehat{H}_2, \infty[$

B. If $g(\widehat{H}_2) < 1/G$ and $g(x_2^*) > 1/G$,there are two solutions in $]\widehat{H}_2, \infty[$

C. If $g(\widehat{H}_2) < 1/G$ and $g(x_2^*) < 1/G$,there is no solution in $]\widehat{H}_2, \infty[$

ii. If $\widehat{H}_2 > x_2^*$

A. If $g(\widehat{H}_2) < 1/G$,there is no solution in $]\widehat{H}_2, \infty[$

B. If $g(\widehat{H}_2) > 1/G$,there is one solution in $]\widehat{H}_2, \infty[$

(b) If $(\alpha+\beta-1)>0$ the graph of $g(x)$ is the following:

	0		∞
$g^{'}(x)$			$+$
$g(x)$	0	$\nearrow$	∞

i. If $g(\widehat{H}_2)<1/G$, there is a unique solution H_3 in $]\widehat{H}_2,\infty[$

ii. If $g(\widehat{H}_2)>1/G$, there is no unique solution in $]\widehat{H}_2,\infty[$

3. the dynamics:

(a) The dynamics in $]H_1,\widehat{H}_2[$ is given by $\psi_1(x)=b\frac{\gamma(1-\alpha)}{(\gamma+1)\lambda}\left[\frac{\alpha\lambda}{p_0(1-\alpha)}\right]^{\alpha}\left[(1+\lambda)x-\overline{c}\right]x^{\alpha+\beta-1}$,wh is an increasing function of x.

Let $f(x)=\left[(1+\lambda)x-\overline{c}\right]x^{\alpha+\beta-1}$

$$f^{'}(x)=x^{\alpha+\beta-2}\left[(\alpha+\beta)\,(1+\lambda)x-(\alpha+\beta-1)\,\overline{c}\right]$$

Let $x^{*}=\frac{(\alpha+\beta-1)\overline{c}}{(\alpha+\beta)(1+\lambda)}$

i. If $(\alpha+\beta-1)<0$, $f^{'}(x)>0$

ii. If $(\alpha+\beta-1)>0$, $x^{*}<H_1$. So $f^{'}(x)>0$

(b) The dynamics in $]\widehat{H}_2,\infty[$ is given by $\psi_2(h_t)=b\left[\frac{\gamma\alpha}{p_0(1+\alpha\gamma)}\right]^{\alpha}(h_t-\overline{c})^{\alpha}h_t^{\beta}$.$\psi_2$ is not necesserally a monotonous function on $]\widehat{H}_2,\infty[$

Let $f(x)=(x-\overline{c})^{\alpha}x^{\beta}$

$$f^{'}(x)=x^{\beta-1}(x-\overline{c})^{\alpha-1}\left((\beta+\alpha)\,x-\beta\overline{c}\right)$$

Let $x^{*}=\frac{\beta\overline{c}}{(\alpha+\beta)}$. We notice that $x^{*}<\overline{c}$; as $\widehat{H}_2>\overline{c}$, we have $f^{'}(x)>0$ for $x\in]\widehat{H}_2,\infty[$.

Lemma 15 *If* $\alpha+\beta-1<0$ *and* $\gamma\,(1-\alpha\,(1+\lambda))-\lambda>0$ *then* $x_1^{*}>x_2^{*}$

Proof of the lemma: Let us assume that $x_1^{*}\leq x_2^{*}$. We then have

$$\begin{aligned}(\alpha+\beta-2) &\geq (\beta-1)\,(1+\lambda) \\ \frac{1-\alpha}{1-\beta} &\leq \lambda\end{aligned} \tag{9}$$

Moreover, as $\gamma\,(1-\alpha\,(1+\lambda))-\lambda>0$, we have

$$1-\alpha\,(1+\lambda)>\frac{\lambda}{\gamma}>\lambda\ (\text{as }\gamma<1) \tag{10}$$

From inequalities 9 and 10,

$$\begin{aligned}(1-\alpha(1+\lambda))(1-\beta) &> 1-\alpha \\ \frac{\alpha-1}{1-\beta} &> \frac{\alpha}{\beta}\lambda\end{aligned}$$

which is impossible as $\alpha - 1 < 0$.

PROOF OF PROPOSITION 8

1. Steady states in $]\overline{H}_1, \overline{H}_2[$ are solutions of $f(x) = F$ where $f(x) = [(1-\tau+\lambda)x - \overline{c}]\, x^{\alpha+\beta-2}$ and $F = \frac{(1+\gamma)[p_0\tau+\lambda]}{b\tau^\alpha\gamma}$

   $$f'(x) = x^{\alpha+\beta-3}\left[(\alpha+\beta-1)(1-\tau+\lambda)x - (\alpha+\beta-2)\,\overline{c}\right]$$

   Let $x^* = \frac{(\alpha+\beta-2)\overline{c}}{(\alpha+\beta-1)(1-\tau+\lambda)}$

   (a) If $(\alpha+\beta-1) > 0$ then $\frac{(\alpha+\beta-1-1)\overline{c}}{(\alpha+\beta-1)(1-\tau+\lambda)} < \overline{H}_1$. The graph of f is the following

	$\overline{H}_1$		∞
$g'(x)$			$+$
$g(x)$	0	$\nearrow$	∞

      i. If $f(\overline{H}_2) < F$ there is no steady state in $]\overline{H}_1, \overline{H}_2[$

      ii. If $f(\overline{H}_2) > F$ there is a unique steady state in $]\overline{H}_1, \overline{H}_2[$

   (b) If $(\alpha+\beta-1) < 0$ then $\frac{(\alpha+\beta-1-1)\overline{c}}{(\alpha+\beta-1)(1-\tau+\lambda)} > \overline{H}_1$. The graph of f is the following

	$\overline{H}_1$		x^*		∞
$f'(x)$		$+$	0		$-$
$f(x)$	0	$\nearrow$	$f(x^*) > 0$	$\searrow$	0

      i. If $x^* > \overline{H}_2$

         A. If $f(\overline{H}_2) < F$ there is no steady state in $]\overline{H}_1, \overline{H}_2[$

         B. If $f(\overline{H}_2) > F$ there is a unique steady state in $]\overline{H}_1, \overline{H}_2[$

      ii. If $x^* < \overline{H}_2$

         A. If $f(\overline{H}_2) > F$ there is one steady state in $]\overline{H}_1, \overline{H}_2[$

B. If $f(\overline{H}_2) < F$ and $f(x^*) < F$ there is no steady state in $]\overline{H}_1, \overline{H}_2[$

C. If $f(\overline{H}_2) < F$ and $f(x^*) > F$ there are two steady states in $]\overline{H}_1, \overline{H}_2[$.

2. If $\left(\frac{1}{b\tau_t^{\alpha}}\right)^{\frac{1}{\alpha+\beta-1}} > H_{2,}$ there is a unique steady state $\left(\frac{1}{b\tau_t^{\alpha}}\right)^{\frac{1}{\alpha+\beta-1}} > \overline{H}_2$.
   If $\left(\frac{1}{b\tau_t^{\alpha}}\right)^{\frac{1}{\alpha+\beta-1}} < H_{2,}$ there is no steady state in $]\overline{H}_2, \infty[$.

3. The dynamics on $]\overline{H}_1, \overline{H}_2[$ depends on $\varkappa_1(x) = b\tau^{\alpha}\left[\frac{\gamma}{(1+\gamma)[p_0\tau+\lambda]}\right][(1-\tau+\lambda)x - \overline{c}]\, x^{\alpha+\beta-1}$.

   Let $f(x) = [(1-\tau+\lambda)x - \overline{c}]\, x^{\alpha+\beta-1}$

   $$f'(x) = x^{\alpha+\beta-2}\left[(\alpha+\beta)(1-\tau+\lambda)x - (\alpha+\beta-1)\overline{c}\right]$$

   Let $x^* = \frac{(\alpha+\beta-1)\overline{c}}{(\alpha+\beta)(1-\tau+\lambda)}$; we notice that $x^* < \overline{H}_1$; this implies that $\varkappa_1$ is an increasing function on $]\overline{H}_1, \overline{H}_2[$.

4. The dynamics on $]\overline{H}_2, \infty[$ depends on $\varkappa_2(x) = b\tau_t^{\alpha} h_t^{\alpha+\beta}$, which is an increasing function.