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ABSTRACT

Federal Unemployment Reinsurance and Local Labor-Market Policies*

Consider a union of atomistic member states, each faced with idiosyncratic business-cycle shocks. Private cross-border risk-sharing is limited, giving a role to a federal unemployment-based transfer scheme. Member states control local labor-market policies, giving rise to a trade-off between moral hazard and insurance. Calibrating the economy to a stylized European Monetary Union, we find notable welfare gains if the federal scheme's payouts take the member states' past unemployment level as a reference point. Member states' control over policies other than unemployment benefits can limit generosity during the transition phase.

JEL Classification: E32, E24, E62

Keywords: unemployment reinsurance, labor-market policy, fiscal federalism, search and matching

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1 Introduction

Fifty years have passed since a milestone was reached in the efforts to bring to life a European economic and monetary union: the 1970 Werner Report. Already at that time an established view was that a monetary union requires some form of federal fiscal capacity. Such capacity would provide countercyclical transfers so as to help stabilize member states' economies in the face of idiosyncratic shocks or to compensate for the welfare loss of fluctuations; a classic reference is Kenen (1969), and more recently Farhi and Werning (2017). At the same time, it is clear that any federal scheme may require elements of accountability if member states retain some authority over local policies; see Persson and Tabellini (1996).

We analyze one particular form of federal fiscal capacity: federal unemployment reinsurance (RI). Such a scheme provides transfers to each member state based on local unemployment rates and, once in place, could work without discretion. We ask how such a scheme should trade off insurance and the incentives to free-ride if member states retain authority over local labor-market policies. Should the scheme, for example, feature thresholds as in the U.S. unemployment insurance system, that is, provide for federal transfers only if unemployment exceeds a certain level? Or should the federal payouts be indexed to past unemployment, another dimension of the U.S. unemployment insurance system?

In a model calibrated to a stylized European monetary union (with fluctuations in economic activity that are of the same size as witnessed in the crisis years), we find that indexation, in particular, allows federal RI to provide both stabilization of the member states' business cycles and insurance.

We model a union of atomistic and *ex-ante* identical member states. At the member-state level, the modeling follows Jung and Kuester (2015). Each member state is subject to idiosyncratic business-cycle risk. There is no private risk-sharing between member-state households, nor can member-state governments borrow from each other. This lack of liquidity and cross-country insurance provides the rationale for a federal fiscal capacity, which we assume can borrow in the international financial market. Labor markets in each member state are characterized by search and matching frictions. This renders unemployment an indicator of the business cycle, but also means that a member state's labor-market policies can affect payments under the federal RI scheme. We show analytically that federal RI affects the local labor-market policy mix in a direction that implies less employment in the long run. Still, a fall in employment is not a sufficient statistic for the welfare effects. This is so because on the one hand federal transfers reduce fluctuations in consumption and employment (and, thus, the severity of cyclical fluctuations themselves) and, on the other hand, because a lack of international insurance means that – absent a federal RI scheme – member states tend to self-insure through too much employment. Some fall in long-run employment, therefore, need not be detrimental. Rather, a welfare assessment will require a quantitative evaluation.

Our quantitative findings are as follows: absent a behavorial response by member states, the optimal federal RI scheme would entail transfers that virtually compensate for all of the member states' fluctuations in national income and would notably help smooth local business cycles. Against this background, we assess the design of federal RI if member states can change local labor-market policies after federal RI is introduced. We have two important results.

First, well-designed federal RI may remain effective even with a behavioral response by member states. For this, we show that it is essential that the federal RI scheme dampens incentives to free-ride the right way. Namely, notable gains from federal RI arise in the longer run when federal payouts are indexed to a member state's past unemployment rates, but not if the scheme only pays in severe recessions. The reason is that the latter type of scheme misses out on the stabilization gains provided by RI in normal times. In the longer run, indexation allows federal RI to bring about half the gains that would prevail absent a response by member states (welfare gains equivalent to an increase of two-tenths of a percent in lifetime consumption and a fall in employment volatility by half).

Second, member states' control over unemployment benefits is not the central element that curbs the potential generosity of federal RI. In our simulations, instead, it is the member states' ability to adjust other labor-market regulation (hiring subsidies and layoff taxes in our modeling) that turns out to have the potential to severely limit the optimal generosity of federal RI. If member states are free to adjust a broad range of labor-market policies, they can exert control over employment without having to distort the insurance they provide to workers; free-riding becomes less costly. This matters for the incentives on the transition path after federal RI is introduced, in particular. Namely, even indexing payouts to past unemployment leaves some incentives to free-ride immediately after federal RI is introduced. The reason for this is simple: past unemployment is a slow-moving indicator and, thus, captures member states' policy choices only with a lag. Accounting for the ensuing incentives to free-ride on the transition path, therefore, the optimal federal RI scheme is less generous. It caps the amount of transfers provided at some level, even for severe recessions (to a maximum transfer of about 1 percent of GDP in our simulations).

Still, even then federal RI may reap up to one-eighth of the welfare gains that federal RI would provide absent a behavioral response by member states.

The class of federal RI schemes we consider is flexible enough to allow for the main characteristics of various schemes proposed in policy circles, such as by Beblavý et al. (2015) or Bènassy-Quéré et al. (2018). For example, it allows for federal payouts to increase linearly in unemployment (such as is implicit in several European member states' own *local* unemployment insurance schemes to date) or for payouts to a member state that rise non-linearly in unemployment (resembling a threshold system as in the U.S. federal unemployment insurance component, for example). We also allow for a further element of accountability, namely, for payouts to depend on the extent to which current unemployment exceeds the member state's past unemployment rate averaged over several years, another element that figures in the federal component of the U.S. unemployment insurance system. We choose the parameters of such schemes so as to obtain federal RI schemes that maximize the *ex-ante* welfare of the union's constituents under the condition that the federal RI budget be balanced in net present value terms, and anticipating the member states' policy response. Payouts from the federal RI scheme are financed through a flat contribution by member states. Member states can adjust the permanent level of labor-market policy instruments once after the federal RI scheme is introduced.

At the member-state level, business-cycle fluctuations arise from productivity shocks. Wage rigidities amplify these shocks and lead to inefficiently high unemployment in recessions. In our modeling, federal RI transfers not only provide consumption insurance in recessions, but they also loosen the member states' government budget constraint and, thereby, the tax burden on firms. Federal RI, therefore, not only provides consumption insurance to households, but it also helps reduce the depth of recessions in the first place.

The rest of the paper proceeds as follows. Next, we put the paper into the context of the literature. Section 2 spells out the model and the member states' and federal government's problems. The same section explains the transmission channel through which federal RI can provide stabilization and shows analytically how federal RI may affect the member states' labor-market policy mix. Section 3 provides the calibration and discusses our numerical implementation. Section 4 derives the quantitative results discussed above. A final section concludes.

Related literature

To the best of our knowledge, ours is the first paper that studies the design of implementable federal unemployment reinsurance schemes in a quantitative business-cycle model, allowing for an optimal response of labor-market policies by member states. The following aims to make clear what we do (and what we do not do).

We introduce a federal unemployment-based transfer scheme in a union of member states that have authority over local labor-market policies. The need for local policies arises from local frictions: idiosyncratic unemployment risk combined with moral hazard by households gives rise to imperfect insurance of the unemployed. Wage rigidity amplifies cyclical responses. Considerations for labor-market policies for single countries are discussed, for example, in Landais et al. (2018), Mitman and Rabinovich (2015), and Jung and Kuester (2015). Birinci and See (2019) focus on economies with self-insurance by households, from which we abstract. Jung and Kuester (2015) assessed the optimal labor-market policy mix over the business cycle for a single country. By way of reference, empirical considerations for such cyclical labor-market policies are discussed in Chodorow-Reich et al. (2018) and Hagedorn et al. (2013) (for benefits). There remains a debate as to what extent countries can, indeed, engage in countercyclical labor-market stabilization policy. Cahuc et al. (2018), for example, find that hiring subsidies are an effective countercyclical stabilization policy amid rigid wages, but only if well-targeted. To steer clear of that debate, the current paper assumes that member states do not (or cannot) engage in countercyclical labor-market stabilization policies of their own. Instead, we focus on the stabilization that federal RI can provide amid permanent behavioral responses by member states.

The current paper's insights are linked to an extensive literature on fiscal federalism; see Alesina and Wacziarg (1999) or Oates (1999) for references to the literature. A central reference for us is Persson and Tabellini (1996), who ask if fiscal risk-sharing can induce local governments to underinvest in programs that alleviate local risk. Theirs is a static setting with two countries and two states of nature per country. One way to think about our paper is as an extension of the literature to a dynamic environment. Dynamics allow us to calibrate the potential quantitative gains from risk-sharing and to separate considerations for the short and the long run. This is important. In particular, we find that accounting for the transition path shortly after federal RI is introduced (the dynamic incentives) can considerably weaken the optimal generosity of federal RI (and, thus, the welfare benefits) if member states have authority over a broad range of labor-market policies.

The current paper abstracts from nominal rigidities and monetary policy. The simplifying view that we take is that monetary policy would remove the aggregate component of shocks, leaving only member-state differences to be addressed. In Farhi and Werning

(2017), instead, for a given fiscal policy at the member-state level, nominal rigidities give rise to aggregate-demand externalities and provide scope for federal fiscal transfers. The current paper is related, but focuses on how the member states' behavioral response shapes the very scope of federal transfers. To show this, we take labor-market rigidities as the central element that amplifies the welfare costs of business cycles. Needless to say that also with nominal rigidities, one could ask which fiscal mix helps overcome the costs.

In terms of solution techniques, we employ fourth-order perturbation to allow for non-linearity in the federal RI scheme. To accomplish this, we extend the moment formulas in Andreasen et al. (2018) to the fourth order and then follow Levintal (2017). While we have not done so, the solution techniques we employ should allow for extensions to more complex environments as well. Nominal rigidities are one case in point. Another would be accommodating the precautionary motives of households as in Ravn and Sterk (forthcoming), from which we currently abstract.

We look at deliberately simple federal transfer mechanisms triggered by unemployment. Our motivation is that more information (such as policy choices) may not be observable, or it may not be possible to sufficiently formalize the use of that information to form part of a contract or law. This setting allows us to handle cyclical economies with endogenous unemployment fluctuations. Another advantage is that we can allow for a wide range of policy choices by the member states. Necessarily, there are downsides as well. Namely, in contrast to the endogenous incomplete-markets literature, such as Atkeson (1991) and Phelan and Townsend (1991), we take a strong form of international market incompleteness as given. And we assume commitment to the federal RI scheme.

Our paper is motivated by empirical evidence that finds that the European Monetary Union, to date, has limited mechanisms for risk-sharing; see, in particular Furceri and Zdzienicka (2015). For the U.S., the literature, instead, finds notable fiscal risk-sharing. Feyrer and Sacerdote (2013), and earlier Asdrubali et al. (1996), estimate that between 13 and 25 cents of every dollar of a state-level income shock are offset by federal fiscal policy. Our paper suggests that federal unemployment reinsurance could go a long way in achieving such risk-sharing in Europe, if accountability is ensured. Our paper also highlights that mechanical calculations of risk-sharing may miss another important benefit of federal RI, namely, that such RI may stabilize the business cycle in the first place (and, thus, reduce the cyclical divergence in incomes).

Our work considers transitory *ex-post* heterogeneity, but abstracts from *ex-ante* heterogeneity. So do Cooper and Kempf (2004). They show in a two-period setting that a scheme of federal unemployment-based transfers can help overturn the Mundellian (1961) argument that monetary unions are a straitjacket. In contrast to us, they take unemployment in each member state as exogenous to the member-state's policy choices, however. Ábrahám et al. (2019) put great effort into measuring labor-market flow rates across the different euro-area member states. They then build a heterogeneous-agent macro model to study the optimal level of federal unemployment benefits, taking the heterogeneity across member states as given. The authors find that a harmonized unemployment insurance scheme is Pareto-improving relative to the *status quo*. In our setting, member states choose an optimal local unemployment insurance system already in autarky. We then ask what the benefits of a federal RI scheme are on top of this. Moyen et al. (2019) consider a heterogeneous monetary union of two countries. In their setting, a federal planner can mandate the cyclical level of benefits in each member state and the member state cannot adjust other labor-market policies. The authors find considerable scope for such an unemployment insurance scheme in smoothing the business cycle. The same is true for Enders and Vespermann (2019), who analyze the stabilizing effect of federal unemployment insurance in a New Keynesian two-country DSGE model, and for Dolls et al. (2018), who do micro simulations for the previous euro-area recession. Also using a two-country DSGE model, Evers (2015) assesses the risk sharing and welfare properties of several exogenously given fiscal risk-sharing schemes. House et al. (2018) study the role of labor migration as a shock absorber in a monetary union, for constant labor-market instruments. Relative to all these papers, we focus on the optimal scope of federal unemployment-based transfers when member states retain and use authority over local policies.

2 The model

There is a federal union that consists of a unit mass of atomistic, *ex-ante* identical member states that are subject to member-state-specific shocks. Member states are marked by subscript i. Member states control their own UI benefit system, layoff taxes, hiring subsidies, and production taxes leveled on domestic firms, adhering to a balanced-budget requirement. There is no international borrowing and lending, nor is there self-insurance by households. A federal UI scheme makes unemployment-dependent transfers to the member-state government.[1] Time is discrete and runs from $t = 0$ to infinity.

[1]At the level of the member state, the model and exposition build extensively on Jung and Kuester (2015).

2.1 The member-state economy

There are three types of agents in each member state: a unit mass of infinitely lived workers, an infinite mass of potential one-worker firms that produce a homogeneous final good, and the member-state government.

2.1.1 Workers

A worker's lifetime utility is given by

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [\mathtt{u}(c_t^i) + \overline{h} \cdot I(\text{not working}_t) - \iota \cdot I(\text{search}_t)] \right\}. \tag{1}$$

E_0 denotes the expectation operator. $\beta \in (0,1)$ is the time-discount factor. The worker draws utility from consumption, c_t. Felicity function $\mathtt{u}(c) : \mathcal{R} \to \mathcal{R}$ is twice continuously differentiable, strictly increasing and concave. I is the indicator function. If not employed, the worker enjoys an additive utility of leisure $\overline{h}$. Search for a job is a 0-1 decision. Workers are differentiated by a utility cost of search, ι, incurred only if the worker searches for a new job. Across both workers and time $\iota \sim F_\iota(0, \sigma_\iota^2)$ is *iid*, where $F_\iota(\cdot,\cdot)$ marks the logistic distribution with mean 0 and variance $\sigma_\iota^2 = \pi\frac{\psi_\iota^2}{3}$, with $\psi_\iota > 0$ and π being the mathematical constant.

Workers own all firms in their member state in equal proportion. Ownership of firms is not traded. Workers cannot self-insure against income fluctuations through saving or borrowing. Letting Π_t^i mark the dividends that the member state's firms pay and w_t^i the wage that an employed worker earns, consumption of the worker is given by

$$\begin{aligned} c_{u,t}^i &:= b^i + \Pi_t^i \quad \text{if unemployed at the beginning of } t, \\ c_{e,t}^i &:= w_t^i + \Pi_t^i \quad \text{if employed at the beginning of } t. \end{aligned} \tag{2}$$

If the worker enters the period unemployed, the worker receives an amount b^i of unemployment benefits. The assumption is that the member state's government cannot observe the search effort of workers. The government, by assumption, conditions payment only on the worker's current employment status. A worker who enters the period employed receives wage income or, if separated, a severance payment equal to the period's wage.

Value of an employed worker

Let ξ_t^i be the separation rate of existing matches. Before separations occur, the value of an employed worker is

$$V^i_{e,t} = \quad \mathtt{u}(c^i_{e,t}) + [1-\xi^i_t]\beta E_t V^i_{e,t+1} + \xi^i_t[V^i_{u,t} - \mathtt{u}(c^i_{u,t})]. \tag{3}$$

The worker consumes $c^i_{e,t}$. With probability $1-\xi^i_t$ the match does not separate and continues into $t+1$. With probability ξ^i_t, instead, the match separates. The worker can immediately start searching for new employment. Therefore, the only difference of the newly-unemployed worker's value to the value of a worker, who was unemployed to start with, is that the separated worker receives the severance payment while the unemployed worker receives unemployment benefits (and, thus, lower consumption). $V^i_{u,t}$ is the value of a worker who starts the period unemployed.

Value of an unemployed worker and search

An unemployed worker decides to search for a job or not. Only those workers will decide to search whose disutility costs of search fall below a state-dependent cutoff value search. We mark this cutoff by $\iota^{s,i}_t$. This cutoff is defined such that, at the cutoff, the utility cost of search just balances with the expected gain from search:

$$\iota^{s,i}_t = f^i_t \, \beta \, E_t \left[\Delta^i_{t+1}\right]. \tag{4}$$

Here $\Delta^i_t = V^i_{e,t} - V^i_{u,t}$ marks the gain from employment and f^i_t marks the job-finding rate. Using the properties of the logistic distribution, the share of unemployed workers who search is given by

$$s^i_t = Prob(\iota \leq \iota^{s,i}_t) = 1/[1+\exp\{-\iota^{s,i}_t/\psi_s\}]. \tag{5}$$

With this, the value of an unemployed worker at the beginning of the period, before the search preference shock has been realized, is given by

$$\begin{aligned} V^i_{u,t} \quad = \quad & \mathtt{u}(c^i_{u,t}) + \overline{h} \\ & + \int_{-\infty}^{\iota^{s,i}_t} dF_\iota(\iota) + s^i_t \left[f^i_t \, \beta E_t V^i_{e,t+1} + [1-f^i_t]\beta E_t V^i_{u,t+1}\right] \\ & + (1-s^i_t)\beta E_t V^i_{u,t+1}. \end{aligned} \tag{6}$$

In the current period, the worker consumes $c^i_{u,t}$ and enjoys utility of leisure $\overline{h}$ (first row). If the worker decides to search (second row), the utility cost is ι. The term with the integral is the expected utility cost of search. With probability f^i_t the searching worker will find a job. In that case, the worker's value at the beginning of the next period will be $V^i_{e,t+1}$.

With probability $(1 - f_t^i)$ the worker remains unemployed in the next period. If the worker does not search (third row), the worker remains unemployed.

2.1.2 Firms

Profits in the firm sector accrue to the workers, all of whom hold an equal amount of shares in the domestic firms. The decisions made by firms are dynamic and involve discounting future profits. We assume that firms discount the future using discount factor $Q_{t,t+s}^i$, where $Q_{t,t+s}^i := \beta \frac{\lambda_{t+s}^i}{\lambda_t^i}$, with λ_t^i being the weighted marginal utility of the workers (the firms' owners):

$$\lambda_t^i := \left[\frac{e_t^i}{\mathrm{u}'(c_{e,t}^i)} + \frac{u_t^i}{\mathrm{u}'(c_{u,t}^i)} \right]^{-1}. \tag{7}$$

This reflects the fact that a mass e_t^i of workers are employed at the beginning of the period and a mass $u_t^i := 1 - e_t^i$ are unemployed.

Firms need a worker to produce output. A firm that enters the period matched to a worker can either produce or separate from the worker. Production entails a firm-specific resource cost, ϵ. This fixed cost is independently and identically distributed across firms and time with distribution function $F_\epsilon(\mu_\epsilon, \sigma_\epsilon^2)$. $F_\epsilon(\cdot,\cdot)$ is the logistic distribution with mean μ_ϵ and variance $\sigma_\epsilon^2 = \pi \frac{\psi_\epsilon^2}{3}$, with $\psi_\epsilon > 0$. One can show that the firm separates from the worker whenever the idiosyncratic cost shock, ϵ, is larger than a state-dependent threshold that we mark by $\epsilon_t^{\xi,i}$, and define later in equation (17). Using the properties of the logistic distribution, conditional on that threshold, the separation rate can be expressed as

$$\xi_t^i = Prob(\epsilon \geq \epsilon_t^{\xi,i}) = 1/[1 + \exp\{(\epsilon_t^{\xi,i} - \mu_\epsilon)/\psi_\epsilon\}]. \tag{8}$$

Ex-ante, namely, before the idiosyncratic cost shock ϵ is realized, the value of a firm that has a worker is given by

$$\begin{aligned} J_t^i \;=\; & -\xi_t^i \left[\tau_\xi^i + w_t^i\right] \\ & - \int_{-\infty}^{\epsilon_t^{\xi,i}} \epsilon \, dF_\epsilon(\epsilon) + (1 - \xi_t^i) \; \left[\exp\{a_t^i\} - w_t^i - \tau_{J,t}^i + E_t Q_{t,t+1}^i J_{t+1}^i\right]. \end{aligned} \tag{9}$$

Upon separation, the firm is mandated to pay a layoff tax τ_ξ^i to the government and a severance payment of a period's wage w_t^i to the worker (first line). If the cost ϵ does not exceed the threshold, the firm will not separate. Rather, the firm will pay the resource cost, and the match will produce (second line). a_t^i is a member-state-specific labor-productivity shock. This is the source of the *ex-post* heterogeneity of member states. The firm produces

$\exp\{a_t^i\}$ units of the good and pays the wage w_t^i to the worker. In addition, the firm pays a production tax $\tau_{J,t}^i$. A match that produces this period continues into the next.
The labor-productivity shock, a_t^i, evolves according to

$$a_t^i = \rho_a \, a_{t-1}^i + \varepsilon_{a,t}^i, \quad \rho_a \in [0, 1), \; \varepsilon_{a,t}^i \sim N(0, \sigma_a^2).$$

An employment-services firm that does not have a worker can post a vacancy. If the firm finds a worker, the worker can start producing from the next period onward. Accounting for subsidies by the member state, the cost to the firm of posting a vacancy is $\kappa_v(1 - \tau_v^i)$. $\kappa_v > 0$ marks a resource cost, and τ_v^i the government's subsidy for hiring. In equilibrium, employment-services firms post vacancies until the after-tax cost of posting a vacancy equals the prospective gains from hiring:

$$\kappa_v(1 - \tau_v^i) = q_t^i E_t \left[Q_{t,t+1}^i J_{t+1}^i \right], \tag{10}$$

where q_t^i is the probability of filling a vacancy.
Let v_t^i be the total mass of vacancies posted in the member state. Matches m_t^i evolve according to a constant-returns matching function:

$$m_t^i = \chi \cdot \left[v_t^i \right]^\gamma \cdot \left[[\xi_t^i e_t^i + u_t^i] s_t^i \right]^{1-\gamma}, \; \gamma \in (0, 1). \tag{11}$$

Here, $\chi > 0$ is match efficiency. The mass of workers who potentially search is $\xi_t^i e_t^i + u_t^i$, with $\xi_t^i e_t^i$ being workers separated at the beginning of the period and u_t^i the mass of workers that enter the period unemployed. s_t^i is the share of those who do actually search. With this, employment evolves according to

$$e_{t+1}^i = [1 - \xi_t^i] \cdot e_t^i + m_t^i. \tag{12}$$

Total production of output is given by

$$y_t^i = e_t^i (1 - \xi_t^i) \exp\{a_t^i\}, \tag{13}$$

where $e_t^i(1 - \xi_t^i)$ is the mass of existing matches that are not separated in t.
For subsequent use, define labor-market tightness as $\theta_t^i := v_t^i / ([\xi_t^i e_t^i + 1 - e_t^i] s_t^i)$, the job-finding rate as $f_t^i := m_t^i / ([\xi_t^i e_t^i + 1 - e_t^i] s_t^i) = \chi [\theta_t^i]^\gamma$, and the job-filling rate as $q_t^i := m_t^i / v_t^i = \chi [\theta_t^i]^{\gamma - 1} = f_t^i / \theta_t^i$.

Dividends

Dividends in each member state arise from the profits generated by the firms that are located in the member state, and are given by

$$\begin{aligned} \Pi_t^i \quad = \quad & -e_t^i \left[\int_{-\infty}^{\epsilon_t^{\xi,i}} \epsilon dF_\epsilon(\epsilon)\right] + e_t^i(1-\xi_t^i)\left[\exp\{a_t^i\} - w_t^i - \tau_{J,t}^i\right] - e_t^i\xi_t^i\left[w_t^i + \tau_\xi^i\right] \\ & - [\kappa_v - \tau_v^i]v_t^i. \end{aligned} \tag{14}$$

2.1.3 Bargaining between firm and worker

At the beginning of the period, matched workers and firms observe the aggregate shock, a_t^i. Conditional on this, and *prior* to observing a match-specific cost shock ϵ_j, firms and workers bargain over the wage and the severance payment as well as over a state-contingent plan for separation. Anticipating that the firm will insure the risk-averse worker against the idiosyncratic risk associated with ϵ_j so that the wage, w_t, is independent of the realization of ϵ_j and that the severance payment equals the wage, firm and worker solve

$$(w_t^i, \epsilon_t^{\xi,i}) = \arg\max_{w_t^i, \epsilon_t^{\xi,i}} (\Delta_t^i)^{1-\eta_t^i}(J_t^i)^{\eta_t^i}, \tag{15}$$

where η_t^i measures the bargaining power of the firm. We shall assume that η_t^i is linked to productivity according to $\eta_t^i = \eta \cdot \exp\{\gamma_w \cdot a_t^i\}$, with $\gamma_w \geq 0$. If $\gamma_w > 0$, the bargaining power of firms is low in recessions and high in booms.

The first-order condition for the wage is as follows

$$(1-\eta_t^i)J_t^i = \eta_t^i \frac{\Delta_t^i}{\mathrm{u}'(c_{e,t}^i)}. \tag{16}$$

This states that after adjusting for the bargaining weights, the value of the firm equals the surplus of the worker from working, expressed in units of consumption when employed.

The first-order condition for the separation cutoff yields

$$\epsilon_t^{\xi,i} \quad = \quad \left[\exp\{a_t^i\} - \tau_{J,t}^i + \tau_\xi^i + E_t Q_{t,t+1}^i J_{t+1}^i\right] + \frac{\beta E_t \Delta_{u,t+1}^i + \psi_s \log(1-s_t^i) - \overline{h}}{\mathrm{u}'(c_{e,t}^i)}. \tag{17}$$

2.1.4 Federal RI scheme and market clearing

As for the federal unemployment reinsurance scheme, let $\mathbf{B}_F\left(u_t^i; u_t^{avg,i}\right)$ mark transfers of final goods from the federal level to the governments of member states. These transfers are conditioned on the member state's current unemployment, u_t^i. They may also be conditioned on a moving index of past unemployment,

$$u_t^{avg,i} := \delta u_{t-1}^{avg,i} + (1-\delta)u_{t-1}^i, \text{with } \delta \in (0,1). \tag{18}$$

All member states are subject to the same structure of the federal RI scheme. Let $\boldsymbol{\tau}_F$ mark a flat, time-independent contribution to the federal RI scheme, paid by each member state. Anticipating that member-state governments do not have access to international borrowing or lending (see Section 2.2.2), goods market clearing in each member state requires that in each of them

$$y_t^i + \mathbf{B}_F\left(u_t^i; u_t^{avg,i}\right) - \boldsymbol{\tau}_F = e_t^i c_{e,t}^i + u_t^i c_{u,t}^i + e_t^i \int_{-\infty}^{\epsilon_t^{\xi,i}} \epsilon \, dF_\epsilon(\epsilon) + \kappa_v v_t^i. \tag{19}$$

The left-hand side has goods produced in the member state plus the net transfers received under the federal RI scheme. In equilibrium, goods are used for consumption (the first two terms on the right-hand side), for production costs, or for vacancy-posting costs. Once markets clear in all the member states, they also clear for the union as a whole.
In terms of accounting, in the calibration later on, we view the resources spent on retaining the match as intermediate goods, such that the definition of GDP is

$$gdp_t^i = y_t^i - e_t^i \int_{-\infty}^{\epsilon_t^{\xi,i}} \epsilon \, dF_\epsilon(\epsilon). \tag{20}$$

Market clearing expressed in units of GDP then is

$$gdp_t^i + \mathbf{B}_F\left(u_t^i; z_t^i\right) - \boldsymbol{\tau}_F = e_t^i c_{e,t}^i + u_t^i c_{u,t}^i + \kappa_v v_t^i. \tag{21}$$

2.2 Government sector

There are two levels of government: the federal level and the local (member-state) level. At the beginning of period $t = 0$, before idiosyncratic shocks to member states have materialized, the federal government can set up a federal RI scheme, knowing the initial distribution of member states in the state space, and anticipating the response by member states and households. Period $t = 0$ is the first period in which the scheme will make payouts and collect contributions. The federal government is a first-mover. The federal RI scheme is implemented in a permanent manner and under full commitment. This choice of timing seems a reasonable first pass for many countries; we believe it is even more so for the European Monetary Union/European Union, where changes to binding agreements often require unanimity. Member-state governments, when choosing their permanent labor-market policies, take the federal RI scheme as given. We describe each level of government in turn.

2.2.1 The federal government's problem

Let μ_t mark the distribution of member states across the possible states of the economy, in period t. Let $\tilde{\mu}_t$ be the induced distribution over the payout-relevant characteristics $(u_t^i, u_t^{avg,i})$. Throughout, we condition our solutions on the initial distribution of member states prior to period $t = 0$. Note that, once one does this, by the law of large numbers both μ_t and $\tilde{\mu}_t$, $t = 0, 1, ...$, are measurable at the beginning of period $t = 0$.[2]

The federal government has access to international borrowing and lending at a fixed gross interest rate $1 + r = 1/\beta$. The federal RI scheme has to be self-financing in the sense that payouts or any debt must be financed completely by the federal RI taxes. Assuming that there is no initial debt, this is the case if

$$\sum_{t=0}^{\infty}(1+r)^{-t}\int\left(\mathbf{B}_F(u_t^i; u_t^{avg,i}) - \boldsymbol{\tau}_F\right)\, d\tilde{\mu}_t = 0. \tag{22}$$

Here, the integral is over the distribution of $(u_t^i, u_t^{avg,i})$ in all member states in the respective period t.

Weighting all households in a member state equally, using (1) and the logistic distribution, after shocks have been realized in period t a member state's utilitarian welfare function can be written as (see Jung and Kuester (2015) for a derivation)

$$W_t^i := E_t \sum_{k=t}^{\infty} \beta^k \left[e_k^i \mathtt{u}(c_{e,k}^i) + u_k^i \mathtt{u}(c_{u,k}^i) + \ (e_k^i \xi_k^i + u_k^i)(\Psi_s(s_k^i) + \overline{h})\right]. \tag{23}$$

The first term is the consumption-related utility of employed workers. The second term is the consumption-related utility of unemployed workers. The third term refers to the value

[2]The position of each member state i in the distribution is random. Since all risk is idiosyncratic, though, and member states are given equal weight in the federal planner's welfare function below, it does not matter which member state is in what position of the distribution.

of leisure and the utility costs of search.[3] The federal government's problem then is to

$$\begin{aligned} \max_{\mathbf{B}_F(\cdot;\cdot),\boldsymbol{\tau}_F} \quad & \int W_0^i \, d\mu_0 \\ \text{s.t.} \quad & \text{member states' policy response (see Section 2.2.2)} \\ & \text{induced law of motion of member states' economies (earlier sections)} \\ & \text{financing constraint (22),} \end{aligned} \tag{24}$$

where the maximization over $\mathbf{B}_F(\cdot;\cdot)$ indicates that the federal government chooses the shape of the payout function of the federal RI scheme. Throughout, we assume that $\mathbf{B}_F(\cdot;\cdot)$ will be continuously differentiable. In choosing the payout function, the federal planner anticipates the response to the scheme by both the member states' governments and by the constituents of each member state. Last, the federal RI scheme is restricted to break even.

2.2.2 The member-state government's problem

The member-state government does not have access to international financial markets, nor does it issue debt to local residents. The member-state government faces the budget constraint

$$e_t^i(1-\xi_t^i)\tau_{J,t}^i + e_t^i\xi_t^i\,\tau_{\xi,t}^i + \mathbf{B}_F\left(u_t^i; u_t^{avg,i}\right) = u_t^i b_t^i + \kappa_v \tau_{v,t}^i v_t^i + \boldsymbol{\tau}_F. \tag{25}$$

The left-hand side shows the revenue from the production and layoff taxes, and the transfers received under the federal RI scheme. The right-hand side has unemployment benefits and vacancy subsidies paid by the member state, as well as the federal RI contribution.

We model the member-state government as a utilitarian Ramsey planner that acts in the interest of its own constituency. It chooses unemployment benefits, Pigouvian layoff taxes, and hiring subsidies (or a subset of the three), and lets taxes on production balance the budget. The exposition below assumes that the member state chooses all labor-market policy instruments. The member-state's problem is analogous when the member state only has access to some of the instruments, a case that we explore in the numerical analysis in Section 4.

The member state sets policies in period 0, before shocks have been realized, but after the federal level has announced the shape of its federal RI scheme. The member state

[3] Here $\Psi_s(s_k^i) := -\psi_s\left[(1-s_k^i)\,\log(1-s_k^i) + s_k^i\log(s_k^i)\right]$. $\Psi_\xi(\xi_k^i)$, which is used further below, is defined in an analogous manner.

chooses the levels of policies that remain in place permanently. Thus, the member-state government's problem is to choose state-and-time-independent labor-market policies (with production taxes balancing the budget) according to

$$\begin{aligned} \max_{\{\tau_v^i, \tau_\xi^i, b^i, \tau_{J,t}^i\}} \quad & \int W_0^i \, d\mu_0 \\ \text{s.t.} \quad & \text{a given federal RI scheme } \mathbf{B}_F, \boldsymbol{\tau}_F \\ & \text{the induced law of motion of the member state's economy (earlier sections)} \\ & \text{the member-state government's budget constraint (25),} \end{aligned} \tag{26}$$

where W_0^i is given by (23). Being atomistic, the member-state planner takes the federal RI scheme as given. Also, there is no strategic interaction with other member states' governments. Each member state's government does anticipate, however, how its choice of labor-market instruments affects the local economy. We model a one-time choice of labor-market instruments, with commitment to these values afterward. $\tau^i_{J,t}$ then moves with the state of the business cycle, so as to clear the government's budget.

2.3 The transmission channel of federal transfers

The model above is highly stylized. So it will be useful to explain the mechanism through which federal RI can provide insurance against aggregate fluctuations in income and stabilize the business cycle. Federal RI transfers, in our modeling, are paid to member-state governments. They loosen the government budget constraint, equation (25). For given labor-market policies in the member state, a federal transfer under the RI scheme thus reduces the tax on production, $\tau^i_{J,t}$. This in turn has two effects. The direct effect is that dividends rise; compare equation (2). By assumption, all households in a member state share equally in the dividends, meaning that federal RI payments are passed through to households in a lump-sum fashion. We consider this to be as neutral an assumption on the distribution as one can make. The second effect is that – with wages being rigid – a cut in production taxes will raise the value of a (prospective) worker to a firm; compare equation (9). Therefore, hiring rises and there will be fewer separations. Federal RI transfers thus induce higher employment and, if provided in recessions, can help stabilize the business cycle.

2.4 The effect of federal RI on member states' choice of policies

The previous paragraph has sketched the transmission of federal RI for fixed member-state labor-market policies. The trade-off at the core of the current paper is that the design of federal RI needs to balance the associated gains in welfare and the distortions that arise from the member states' behavioral response to the introduction of this very scheme. The current section provides analytical intuition for that response. For better readability, throughout this section we focus on the steady state. The steady state is symmetric; so steady-state values carry neither a superscript for the country nor a time index. We consider a member state that chooses the entire labor-market policy mix (local UI benefits, layoff taxes, and hiring subsidies). To show the intuition most clearly, we focus on a federal RI scheme that is conditioned on current unemployment in the member state only.

Proposition 1 *Consider the economy described in Section 2. Let* $\Omega := \frac{\eta}{\gamma}\frac{1-\gamma}{1-\eta}$ *be the Hosios measure of search externalities and* $\zeta = \frac{\psi_s}{f(1-s)} \frac{1-e}{[\xi e+(1-e)]} \frac{\mathrm{u}'(c_u)-\mathrm{u}'(c_e)}{\mathrm{u}'(c_u)\mathrm{u}'(c_e)}$ *be a measure of tension between moral hazard and insurance of the* unemployed *in each member state. Suppose there is a given federal RI scheme that is characterized by* $\mathbf{B}_F(u_t^i)$ *and* $\boldsymbol{\tau}_F$*. Let* $\mathbf{B}'_F(u_t^i)$ *mark the first derivative of the federal transfer function. Focus on the steady state. The following labor-market policies and taxes implement the allocation that the member-state planner chooses*

$$\tau_v = [1-\Omega] + \frac{\eta}{1-\eta}\frac{\zeta}{\kappa_v \frac{\theta}{f}}, \tag{27}$$

$$\tau_\xi = \tau_J + \tau_v \kappa_v \frac{\theta}{f} + \zeta(1-sf) - \mathbf{B}'_F(u), \tag{28}$$

$$b = \frac{(1-\beta)}{\beta}\tau_v \kappa_v \frac{\theta}{f} e + \zeta e \frac{[1-\beta(1-sf)(1-\xi)]}{\beta} + [\mathbf{B}_F(u) - \boldsymbol{\tau}_F] + e\mathbf{B}'_F(u), \tag{29}$$

$$\tau_J = \frac{1-e}{e}[b - \zeta s f] - \frac{\mathbf{B}_F(u) - \boldsymbol{\tau}_F}{e}, \tag{30}$$

Proof. *The proof is an extension of the results in Jung and Kuester (2015) to the case of federal unemployment reinsurance. The derivations are available upon request.* ■

Two elements of the federal RI scheme figure prominently. First are the steady-state net transfers per period, $\mathbf{B}_F(u) - \boldsymbol{\tau}_F$. All else equal, the higher the net transfers, the more fiscal space there is for the member state and the higher will be the unemployment benefits b and the lower are taxes on firms τ_J. The effect of this on employment is ambiguous. The second element that figures prominently is the generosity of the federal RI scheme *at the margin*, $\mathbf{B}'_F(u)$. The more generous federal RI is at the margin, the more generous are the unemployment benefits the member state will choose (see equation (29)), and the less stringent will be the layoff restrictions (equation (28)). A federal RI scheme, therefore, interacts not only with the member-state's choice of unemployment benefits (b), but also with the wider labor-market policy mix that the member state considers optimal. The implications can be seen more clearly still under the somewhat less general conditions of the following proposition.

Proposition 2 *Consider the same conditions as in Proposition 1. Define the average duration of an unemployment spell as* $D \equiv \frac{1}{sf}$, *and the average time that an unemployed worker receives unemployment benefits* $D_2 \equiv D - 1$.[4] *Define the elasticity of the latter with respect to unemployment benefits as* $\epsilon_{D_2,b} = \frac{D}{D_2}\frac{f(1-s)}{\psi_s}$. *Assume* $\beta \to 1$, *that utility is logarithmic,* $\mathtt{u}(c) = \log(c)$, *and the Hosios condition* $(\Omega = 1)$ *holds. Then the following characterize the policies that implement the allocation that the member-state planner chooses*

$$\begin{aligned}
\frac{b}{w} &= \frac{1}{1+D\epsilon_{D_2,b}} + \frac{D\epsilon_{D_2,b}}{1+D\epsilon_{D_2,b}}\left[\frac{\mathbf{B}_F(u)-\boldsymbol{\tau}_F}{w} + e\frac{\mathbf{B}'_F(u)}{w}\right], && (31)\\
\frac{\tau_\xi}{w} &= \frac{\frac{D}{1-\eta}-1}{1+D\epsilon_{D_2,b}} - \frac{(D\epsilon_{D_2,b}+\frac{D}{1-\eta})}{1+D\epsilon_{D_2,b}}\left[\frac{\mathbf{B}_F(u)-\boldsymbol{\tau}_F}{w} + e\frac{\mathbf{B}'_F(u)}{w}\right], && (32)\\
\kappa_v\frac{\theta}{f}\frac{\tau_v}{w} &= \frac{\eta}{1-\eta}\frac{D}{1+D\epsilon_{D_2,b}} - \frac{\eta}{1-\eta}\frac{D}{1+D\epsilon_{D_2,b}}\left[\frac{\mathbf{B}_F(u)-\boldsymbol{\tau}_F}{w} + e\frac{\mathbf{B}'_F(u)}{w}\right], && (33)\\
\frac{\tau_J}{w} &= - \left[\frac{\mathbf{B}_F(u)-\tau_F}{w} - u\frac{\mathbf{B}'_F(u)}{w}\right]. && (34)
\end{aligned}$$

autarky (first terms) — *federal RI* (remaining terms)

Proof. *The proof is a special case of Proposition 1.* ■

In each of the equations of Proposition 2, the terms labeled "autarky" are those that would appear absent a federal RI scheme, while "federal RI" marks the terms introduced by the RI scheme. This paper's contribution is to discuss how federal RI shapes member states' incentives. We will, therefore, be rather brief when discussing the member states' choice of instruments in the absence of a federal RI scheme. Absent a federal RI scheme, the steady-state replacement rate $\frac{b}{w}$ is given by a version of the closed-economy Baily-Chetty

[4]These differ because the first period of joblessness is covered by severance payments.

formula (Baily 1978 and Chetty 2006) (see equation (31)), which aligns the private gains to a worker that are associated with an increase in benefits with the social costs. Layoff taxes make domestic firms internalize the *domestic* fiscal costs of layoffs, namely, the payment of unemployment benefits over a typical unemployment spell and the additional cost of hiring subsidies; see equation (32). The latter, in turn, make firms and workers internalize the search externality; see equation (33). Absent a federal RI scheme, production taxes are zero. For a much more detailed discussion of the case without federal RI and more intuition, the reader should see Jung and Kuester (2015).

The contribution of the current paper concerns the terms in square brackets instead. These show how the federal RI scheme affects the member states' choice of labor-market instruments in the long run. Once again, there are two effects. One effect pertains to the net transfers received, $\mathbf{B}_F(u)-\boldsymbol{\tau}_F$. To the extent that net transfers received are positive in the long run, the federal RI scheme allows member states to cut production taxes. This will make hiring more attractive for firms, necessitating fewer hiring subsidies and lower layoff taxes, and allowing for somewhat more generous unemployment benefits; see equations (34) through (31), in that order. Since these different policy changes have offsetting effects on employment, *ex-ante* it is not clear if positive federal transfers in the long run raise or reduce employment.

The second effect of federal RI is less ambiguous, though. Namely, the federal RI scheme not only has direct pecuniary effects, but it also affects member states' optimal labor-market policies at the margin. These are the terms $\mathbf{B}'_F(u)$ in Proposition 2. Provided that higher unemployment in the member state means higher federal RI transfers, that is, if $\mathbf{B}'_F(u) > 0$, the effects of federal RI at the margin point unanimously toward federal RI lowering employment. The member-state planner raises the replacement rate when the marginal payout by the federal RI scheme is positive. Indeed, the member-state planner will raise the replacement rate by the more, the more elastic search is with respect to benefits (the larger $\epsilon_{D_2,b}$) and the larger is the pool of employed workers (e) that – through higher benefits – could be moved into a federally cushioned unemployment spell; see equation (31). In the same vein, the member state will also tend to opt for lower layoff taxes, (32), and expend less on hiring subsidies (33). In combination, these adjustments mean that the member state has funding needs. The production tax, τ_J, will therefore need to rise. Note that all of these changes that are triggered by the incentives at the margin point toward less employment. Taken together, therefore, the proposition suggests that through the effects at the margin, federal RI can induce member states to implement less employment-friendly policies.

The proposition is also suggestive of ways that the federal government could curb an adverse impact of federal RI on long-run unemployment (if it wants to). These ways work through the two terms in the square brackets.

One approach works through the first term, the pecuniary effects. Provided that long-run employment in the member states rises with average payouts, long-run employment will rise if the federal RI scheme is pre-funded. The federal RI schemes that we look at have to break even in net present value; recall equation (22). A federal RI scheme can break even and have positive net payouts in the long run, if its payouts are back-loaded.

The other approach works through the marginal effects of federal RI, the second term in the square brackets. The federal government can curb the effect of federal RI on employment if it implements a scheme in which the marginal effects are small in regions of the state space that the economy visits frequently. Proposition 2 shows the case when transfers are not indexed to a member state's past average unemployment rate. In that case, the marginal incentives can be curbed by implementing a federal RI scheme in which $\mathbf{B}'_F(\cdot) \approx 0$ close to the steady state. This is, indeed, what we find in our numerical analysis in Section 4: unless federal RI is indexed to past average unemployment, the optimal federal RI scheme over a range of unemployment is rather unresponsive to changes in the member state's unemployment rate.

Alternatively, one may try to reduce the long-run marginal effects by indexing transfers in a way such that persistent increases in unemployment do not increase payoffs for the member state. This is what a federal RI scheme achieves in which payouts depend on both current unemployment and an average of past unemployment, that is, a scheme that is indexed to past unemployment. In the long run $u^{avg} = u$. So, in the long run, $\frac{d}{du}\mathbf{B}_F(u, u^{avg}) = \frac{\partial}{\partial u}\mathbf{B}_F(u, u^{avg}) + \frac{\partial}{\partial u^{avg}}\mathbf{B}_F(u, u^{avg})$, where the scheme can be designed such that the two terms on the right-hand side exactly cancel. By design, such a scheme with indexation would take care of the effects on the margin. Note, though, that the propositions above refer to the long run only. Even a scheme that manages to tame long-run incentives may not be able to fully do so on the transition path. We revisit those results in the numerical analysis in Section 4 later.

Last, it is intriguing to observe that federal RI matters for member states' policy choices even if federal RI does not affect the generosity of local unemployment benefits. To see this in Proposition 2, suppose that unemployed workers' search intensity for a new job does not depend on the workers' outside option. In other words, suppose that search is inelastic,

$\epsilon_{D_2,b} = 0$. In this case, member states would grant full consumption insurance, setting $b/w = 1$. This is so irrespective of the design of the federal RI system. Still, federal RI will affect the rest of the labor-market policy mix. The reason is that the federal RI scheme still has pecuniary effects and can change the member state's marginal costs of having a worker unemployed. If the generosity of federal RI increases at the margin, for example, the member-state government will reduce the hiring subsidy, reduce the layoff tax, and raise the production tax. All of this will induce lower employment. Indeed, numerically, we find later that the ability of the member state to adjust its other labor-market policies affects the federal RI scheme at least as much as the member state's ability to adjust local unemployment benefits. As a step toward this quantitative assessment, the next section presents the calibration. Thereafter, Section 4 will, finally, present the quantitative results.

3 Calibration and computation

This section calibrates the model to a stylized European Monetary Union (EMU henceforth). We wish to be clear: As of the time of writing, there is notable heterogeneity across EMU member states. Rather than taking a stand on the extent to which this heterogeneity is exogenous or due to policy choices, however, we deliberately abstract from *ex-ante* heterogeneity altogether. Rather, we wish to ask under which circumstances a federal RI scheme would be expected to provide notable welfare gains in a union of *ex-ante* identical generic member states, and what shape such a federal RI scheme should take in light of the possible behavorial responses by member states.

3.1 Calibration

We assume log utility $\mathtt{u}(c) = \log c$. One period in the model is a month. Our aim is to parameterize the model so as to replicate first and second moments of the data for a generic member state of the EMU (with the 19 member states as of 2019Q4). The sample period is 1995Q1 to 2019Q4. This includes both tranquil periods and the deep recessions during and after the financial and debt crises.

In regard to authority over policies, we target the *status quo*. There is no federal RI. Next, our understanding is that, today, in most member states the labor-market policy mix is kept constant over the business cycle. In the baseline, therefore, member states unilaterally choose a labor-market policy mix (that is, the replacement rate, the layoff tax, and the hiring subsidy) that is fixed over the business cycle.

Under those assumptions, we calibrate model parameters to match average fluctuations and long-run moments for an "average" euro-area country. Toward this end, we obtain country-level data for 14 euro-area member states from Eurostat.[5] Of course, there are endlessly different ways of assigning the movements in variables to the trend or cycle or of assigning fluctuations to the individual country or to the aggregate level. We make two judicious choices here. First, in extracting the cyclical component for each time series, we apply a linear trend (or, if possible, simply demean). We do so such that the drop in GDP in several member states after 2008 and the commensurate rise in unemployment are left as part of the cyclical component of the time series. Second, we assign all fluctuations in the data to country-specific shocks, rather than to the union level. We calculate moments for selected time series and calibrate the member state's economy in our model to a population-weighted average of these moments. We treat the parameters that emerge from the calibration as structural, that is, leave them unchanged in the policy experiments that we conduct later.

Second moments of the data. The business-cycle properties of the data are reported in Table 1. All data are reported at a quarterly frequency. Their model counterparts

Table 1: Business-cycle properties of the data

		y	c	$lprod$	e	$urate$	w
Stand. dev.		3.87	3.45	1.96	3.04	26.16	1.90
Autocorr.		0.96	0.96	0.93	0.99	0.99	0.92
Correlations	y	1.00	0.84	0.57	0.81	-0.61	0.29
	c	-	1.00	0.50	0.66	-0.50	0.27
	$lprod$	-	-	1.00	0.15	-0.14	0.36
	e	-	-	-	1.00	-0.74	0.28
	$urate$	-	-	-	-	1.00	-0.10

Notes: Summary statistics of the data (quarterly). Series are labeled like their counterparts in the model or as described in the text. All data are quarterly aggregates, in logs (including the unemployment rate), and multiplied by 100. We report the cyclical component after applying a linear trend. The exception is the log unemployment rate, which we demean only. Entries can be interpreted as percent deviation from the steady state. The first block reports the standard deviations and autocorrelations. The second block reports cross-correlations of time series within the typical country. The sample is 1995Q1 to 2019Q4. All entries are population-weighted averages of member-state-level moments.

[5]The EA14: Belgium, Germany, Ireland, Greece, Spain, France, Italy, Cyprus, Luxembourg, Malta, Netherlands, Austria, Portugal, and Finland.

are quarterly averages of the monthly observations in the model. The data are seasonally adjusted. The table reports percentage deviations from a linear trend of 100 times the log series (the log unemployment rate is only demeaned).
The first block of the table reports the standard deviation and first-order autocorrelation; the second block reports the cross-correlation of the main aggregates at the country level. All series are from Eurostat. Output y is real gross domestic product (chain-linked volumes). Consumption c is the consumption by households and non-profits divided by the GDP deflator. Labor productivity, $lprod := \frac{y}{e(1-\xi)}$, is measured as our series of GDP divided by employment (heads). The unemployment rate, *urate*, is taken directly from Eurostat. The model counterpart of the unemployment rate is $urate := (e\xi + u)s/[(e\xi + u)s + e(1-\xi)]$ (the mass of non-employed workers who search divided by the labor force). The counterpart of the model's wage, w, is taken to be the ratio of wages and salaries from the national accounts per employee and deflated by the GDP price index.

Targets and parameters. Three of the model's parameters are directly linked to the business cycle: the standard deviation of the productivity shock, σ_a, the dispersion of the continuation costs, ψ_ϵ, and the wage rigidity parameter γ_w. We choose these so as to bring the model as close as possible to matching three business-cycle targets: the standard deviation of measured labor productivity, the standard deviation of the unemployment rate, and the relative standard deviation of the job-finding and separation rate. In our calibration the separation rate is 60 percent as volatile as the job-finding rate, in line with the findings for European OECD countries in Elsby et al. (2013). The other parameters are chosen directly based on outside evidence or using targets for the steady state of the model. The calibrated parameter values are summarized in Table 2. The monthly discount factor, β, equals .996, a customary value. In order to match an average unemployment rate (*urate*) of 9.5 percent, we adjust parameter $\overline{h}$ such that the value of leisure is $\Psi_s(s) + \overline{h} = 0.52$, or 91 percent of the wage. We set $\psi_s = 0.04$ with a view toward matching the micro-elasticity of unemployment with respect to benefits. The value chosen here implies an elasticity of the average duration of unemployment with respect to UI benefits of 0.8, in line with micro estimates such as Meyer (1990). The vacancy posting cost of $\kappa_v = 0.86$ replicates the EMU average monthly job-finding rate of 7.5 percent derived from Elsby et al. (2013). For reference, this gives an average cost per hire net of the hiring subsidy, $\frac{v\kappa_v(1-\tau_v)}{m}$, of a little less than one monthly wage, in line with estimates of recruiting costs (Silva and Toledo,

Table 2: Parameters for the baseline

	description	value	target
Preferences			
β	time–discount factor	0.996	putative real rate of 4% p.a.
$\Psi_s(s) + \bar{h}$	value of leisure.	0.52	st.-st. u rate of 9.5 %
ψ_s	dispers. search cost	0.04	micro-elasticity, Meyer (1990).
Vacancies and matching			
κ_v	vac. posting cost	0.86	EMU avg. monthly job-finding rate.
γ	match elasti. wrt v	0.30	Petrongolo and Pissarides (2001).
χ	match-efficiency	0.12	qtrly job fill rate 71%, den Haan et al. (2000).
Wages			
η	firms' st.-st. barg. p.	0.30	Hosios condition.
γ_w	cyclic. barg. power	13.33	unemployment volatility.
Production and layoffs			
μ_ϵ	mean idios. cost	0.28	share of depreciation in GDP of 20%.
ψ_ϵ	dispers. cost shock	1.74	rel. vola. job-f., sep. rate, Elsby et al. (2013).
ρ_a	AR(1) prod. shock	0.98	qtrly persistence of prod. shock of 0.96.
$\sigma_a \cdot 100$	std. dev.	0.51	standard deviation of measured *lprod*.

Notes: The table reports the calibrated parameter values in the baseline economy.

2009). We set the elasticity of the matching function with respect to vacancies to $\gamma = .3$, which is within the range of estimates deemed reasonable by Petrongolo and Pissarides (2001). The matching-efficiency parameter is set to $\chi = .12$ so as to match a quarterly job-filling rate of 71 percent. We take the latter target from den Haan et al. (2000).

The bargaining power of firms in the steady state is set to $\eta = 0.3$, with an eye on the Hosios (1990) condition. Parameter γ_w governs the rigidity of wages with respect to fluctuations in productivity. We set this to $\gamma_w = 13.33$ to match the variability of unemployment. This implies that for a 1 percent negative productivity shock, the bargaining power of firms falls by 13.33 percent, from a steady-state value of .3 to .26.

The average idiosyncratic cost of retaining a match is set to $\mu_\epsilon = .28$. This parameter governs the average costs of continuing a match. We set the parameter such that in the steady state GDP equals output. Next, we set the dispersion parameter for the idiosyncratic cost shock to $\psi_\epsilon = 1.74$, with a view toward matching the relative volatility of job-finding and job-separation rates in Elsby et al. (2013). Last, we set the serial correlation of the productivity shock to $\rho_a = 0.98$. This translates into a quarterly persistence of the productivity shock of 0.94, within the range of values entertained in the literature. The standard deviation of the shock is set to $\sigma_a = 0.0051$, with an eye on the business-cycle properties (standard deviations) of the model, as discussed above.

Implied business-cycle statistics of the model. Table 3 reports business-cycle statistics for the calibrated model. The calibrated model matches the data reasonably well (compare to Table 1). GDP is about twice as volatile as productivity. The log unemployment rate is about six times as volatile as GDP.

Table 3: Business-cycle properties of the model

		gdp	*c*	*lprod*	*e*	*urate*	*w*	*f*	ξ
Standard dev.		4.70	3.89	1.96	2.90	26.16	2.06	18.77	12.52
Autocorr.		0.98	0.99	0.93	0.99	0.99	0.99	0.96	0.98
Correlations	*y*	1.00	0.99	0.92	0.97	-0.96	0.99	0.98	-1.00
	c	-	1.00	0.86	1.00	-0.99	1.00	0.94	-0.98
	lprod	-	-	1.00	0.80	-0.77	0.87	0.98	-0.93
	e	-	-	-	1.00	-1.00	0.99	0.90	-0.96
	urate	-	-	-	-	1.00	-0.99	-0.88	0.95
	w	-	-	-	-	-	1.00	0.95	-0.99
	f	-	-	-	-	-	-	1.00	-0.99

Notes: Second moments in the model. All data are quarterly aggregates, in logs and multiplied by 100 in order to express them in percent deviation from the steady state. Note: the series for the unemployment rate is in logs as well. The first row reports the standard deviation, the next row the autocorrelation, followed by contemporaneous correlations. Based on a first-order approximation of the model.

Implied steady state. Table 4 reports selected steady-state values for the baseline, including the labor-market policies that the member state chooses. The optimal replacement rate in the steady state (b/w) is 52 percent, a reasonable value for the euro area (compare Christoffel et al. 2009). The optimal vacancy subsidy is $\tau_v = 0.80$. This amounts to a subsidy per actual hire of roughly two and a half monthly wages. The optimal layoff tax equals approximately 9 monthly wages, reasonable given the long average duration of unemployment spells in the EMU and the corresponding fiscal costs.

Impulse responses. Figure 2 on page 29 shows impulse responses to a negative one-standard-deviation productivity shock for the baseline calibration (dashed blue lines labeled "autarky"). To repeat, the baseline does not feature a federal RI system; hence, there are no federal transfers. Benefits b, hiring subsidies τ_v, and layoff taxes τ_ξ are kept constant.

Table 4: Steady-state values

Labor-market policy			Output and consumption		
b	local UI benefits	0.38	gdp, y	GDP and output	0.72
τ_v	vacancy posting subsidy	0.80	c_e	consumption employed	0.73
τ_ξ	layoff tax	6.39	c_u	consumption unemployed	0.38
Labor market			Other variables		
ξ	separation rate	0.008	Π	dividends	0.002
f	job-finding rate	0.078	Δ	gain from employment	1.65
s	search intensity	0.97	J	value of employ.-serv. firm	0.52
e	employment	0.91	τ_J	tax on firms	0.002
u	unemployment	0.0907			
$urate$	unemployment rate	0.095			

Notes: Selected steady-state values for the baseline economy.

By construction of the impulse responses, productivity a_t falls by about half a percent on impact and then gradually recovers (not shown). This directly translates into lower GDP. The fall in GDP is amplified and propagated further by the labor-market response. Wages are rigid and fall only about half as much as productivity. Employment therefore falls (the unemployment rate rises by 0.3 percentage point). The monthly job-finding rate falls by about 0.3 percentage point, the separation rate rises by 0.015 percentage point, and the search intensity of the unemployed falls. With the labor-market policies fixed, higher separations and less recruiting mean that the government earns layoff-tax revenue and saves on hiring subsidies. In spite of the recession, the production tax, therefore, initially falls by 0.3 percent. This response is short-lived, however, and eventually production taxes rise.

3.2 Approximating and evaluating the federal RI scheme

Our paper starts from a setting in which federal RI is beneficial in the first place, and then asks how much of these benefits remain when accounting for member states' behavioral responses to the introduction of the very scheme. In particular, in Section 2.4 we have shown analytically that a federal RI scheme can have first-order effects on member states' policy instruments. Federal RI can, therefore, have first-order effects on the employment level in each member state. Therefore, the class of potentially successful federal RI schemes cannot simply be a linear function of member-state unemployment.

The federal RI schemes that we consider allow for some control of the behavioral response by member states through the shape of the marginal payout. As discussed in Section 2.4, a federal scheme that has flat marginal payouts at or close to the steady state (where the member-state economy spends most of its time) and turns more positive only at high-enough unemployment rates does so, essentially resembling a threshold system. For that reason, as one scenario, we allow for payouts of the form $\mathbf{B}_F(u_t^i)$, where we approximate function $\mathbf{B}_F(\cdot)$ by a fourth-order Chebyshev polynomial. Unless noted otherwise, the outer nodes are fixed at plus/minus two standard deviations of an unemployment increase (measured using the member states' economy in autarky). At the four fixed Chebyshev nodes, let ϕ mark the parameters of the polynomial. We parameterize the amount of transfers by values $\phi = [\phi_1, \phi_2, \phi_3, \phi_4]' \in \mathbb{R}$. Different choices for the parameters allow for a wide range of functional forms of the payouts under the federal RI scheme, including a linear payout function, or payouts that resemble said threshold schemes. At the same time, and importantly for the techniques we use to solve the model, this class of payout schemes always remains differentiable.

Another way of controlling the marginal effects of federal RI in the long run is that the argument of the payouts under the federal RI scheme is not the member state's current unemployment rate, but the *difference* between actual unemployment and a long-term average of unemployment, so that $\mathbf{B}_F(u_t^i; u_t^{avg,i}) = \mathbf{B}_F(u_t^i - u_t^{avg,i})$. This form of indexing payouts to past unemployment makes sure that a permanenent increase in a member state's unemployment is not associated with a permanent increase in transfers. Here, too, we allow for non-linear functional forms in the same way as outlined in the previous paragraph. The one choice that has to be made is over what horizon do we measure average past unemployment. We choose a geometric average as in equation (18) and set the decay parameter δ to compromise between allowing for persistent transfers in persistent recessions on the one hand, and having a measure that eventually does respond to changes in local unemployment rates. For the rest of the paper, we choose ($\delta = 1 - 1/120$), which implies a half-life of about seven years, the length of a typical full business cycle.

The model has too many states to solve it repeatedly by global methods. Instead, throughout we rely on perturbation methods. The polynomials for $\mathbf{B}_F(\cdot)$ are flexible enough to allow for notable non-linearity (such as a scheme that is flat in the left tail and has threshold-like behavior in the right tail). Since such asymmetries in $\mathbf{B}_F$ are important, we need to make sure that the asymmetries feed through to how we solve the model and evaluate welfare. In line with this, we solve the model through a fourth-order perturbation with pruning. Toward evaluating welfare, we extend the moment formulas in Andreasen et al. (2018) to fourth order; see Appendix A. For solving the model, we rely on the routines by Levintal (2017).

Last, we provide details on how we numerically address the maximization problems of the member states and the federal government. Our aim is to obviate the need for Monte Carlo evaluations (such as drawing from the ergodic distribution) when evaluating welfare. Rather, we wish to have numerical evaluations of both the federal government's and the member states' objective functions that become available in closed form as a by-product of solving the model by perturbation. Below, we present two different types of scenarios: exercises that focus on the long run only, and exercises that include the transition path after the introduction of the federal RI scheme. In both of these, our goal is to find an equilibrium in which the federal authority chooses the shape of the RI scheme so as to maximize the expected welfare of member-state households, balancing the federal budget. In choosing, the federal authority anticipates the member states' optimal self-interested policy choices.

When we focus on the long-run incentives only (which is the case in Sections 4.1 and 4.2 below), we evaluate the federal government's and the member states' objective functions conditional on the long-run mean under the respective mix of policies. Note that the means before and after the introduction of the federal RI scheme can differ. So, clearly, this leaves out the question of what happens on the transition path.

When solving for the optimal federal RI including the transition phase (in Section 4.3 below), we condition the welfare evaluations with and without federal RI on all member states entering period $t = 0$ (the first period) from the same state. We assume that the state that all member states start from, both in evaluating member-state policies and in searching for the optimal federal RI scheme, is the non-stochastic steady state implied by our calibration absent federal RI (Table 4). To be clear, while there is no *ex-ante* heterogeneity by this design, *ex-post* there will be.

4 Optimal federal unemployment reinsurance

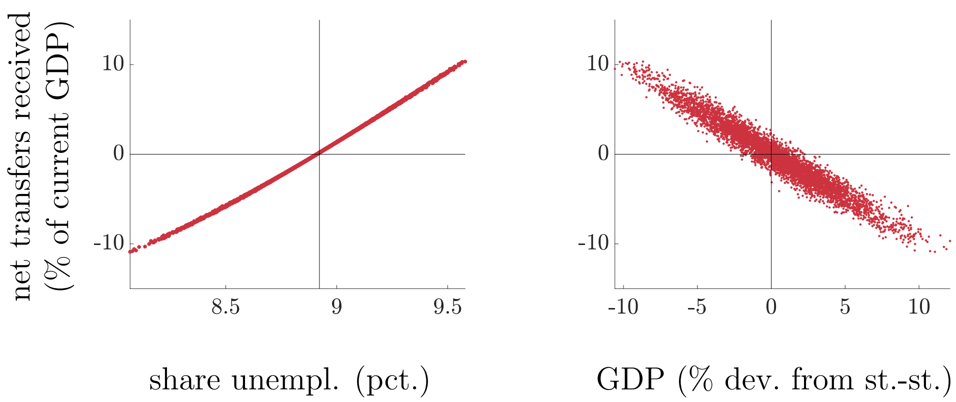
We now turn to analyzing the shape and scope of an *ex-ante* optimal federal RI scheme for our calibrated model of the euro area. As a point of departure, we focus on the long run. We first illustrate the potential scope for federal unemployment reinsurance absent a behavioral response by member states, and then illustrate how member states' incentives shape the scope. Thereafter, we turn to a dynamic perspective. That is, a perspective that explicitly accounts for the incentives on the transition path. Appendix C presents the values of the labor-market policy instruments for each scenario.

4.1 No response by member states: Generous federal RI transfers

To set the stage, we want to know how generous federal unemployment reinsurance should be, abstracting from transition dynamics and from a response by the member state. That is, we hold member states' labor-market policies fixed. In order to compute the optimal federal RI scheme, in this section we take a long-run perspective. The optimal federal RI is computed as if the economy were immediately to jump to the new stochastic steady state. We choose the federal RI that maximizes the unconditional mean of welfare.

Figure 1 plots the net transfers that result under the optimal federal RI scheme, against unemployment (left panel) and GDP (right panel); results do not depend on whether there is indexation to past unemployment rates, so we only show one case here. The baseline economy is characterized by cyclical risk at the member-state level amid financial autarky. That is, the member state is both imperfectly insured and lacks the liquidity to smooth domestic consumption in the face of shocks. Absent a member state's response, these two elements shape the optimal long-run federal RI scheme. The optimal scheme is roughly linear in unemployment (left panel). And it is also generous: the implied transfers make up for virtually all of the output lost in recessions (right panel).

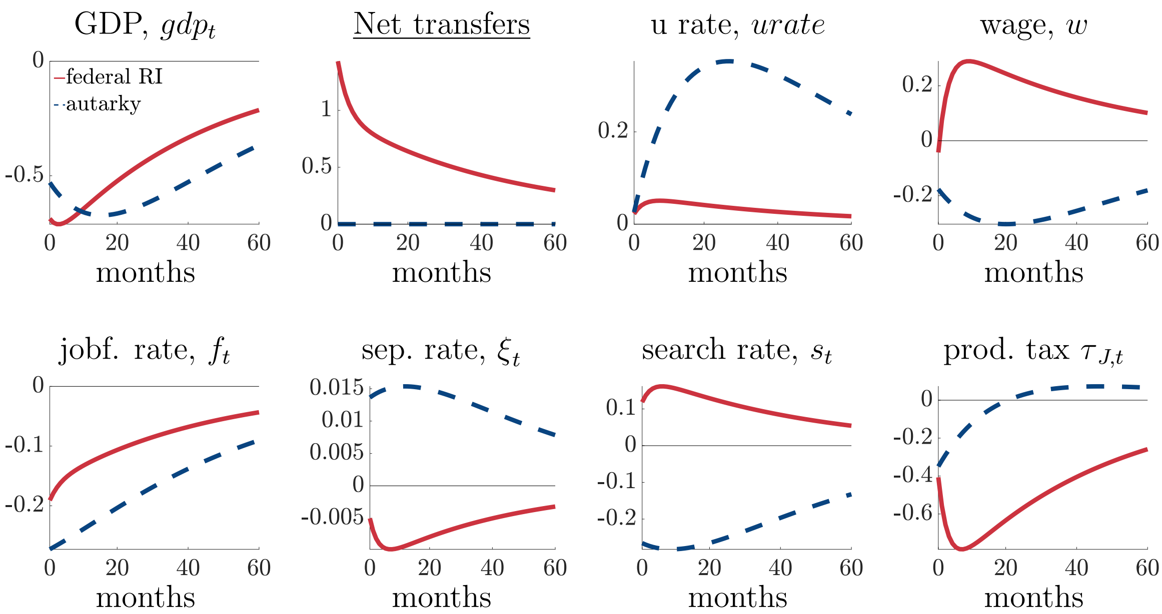
Figure 1: Net transfers received – fixed local policies



Notes: Net transfers received from the rest of the union $\mathbf{B}_F - \boldsymbol{\tau}_F$ (y-axis) against share of unemployed workers (x-axis, left panel) or percent deviation of GDP from the steady state (x-axis, right panel). Based on simulations under the optimal federal RI scheme for the case of fixed policies.

What is more, the optimal federal RI scheme not only stabilizes consumption but it also stabilizes the business cycle itself. Whereas in autarky a 10 percent drop in GDP would be associated roughly with a 6-percentage-point rise in unemployment, with federal RI unemployment rises by only 0.7 percentage point (see the bounds in the left panel of Figure 1).

Figure 2: Impulse responses, optimal federal RI – fixed local policies



Notes: Impulse responses to a one-standard-deviation productivity shock. Shown is the case of autarky (dashed blue lines) and the case with optimal federal RI (solid red lines). Impulse responses are derived under the assumption that the local policy instruments do not react at all to the introduction of a federal RI scheme. All variables are expressed in terms of percent deviation from the steady state (a "1" meaning the variable is 1% above the steady-state level), except for the net transfers received and production tax, which are expressed in percent of GDP.

Figure 2 illustrates the mechanism at work. The figure shows impulse responses to a recessionary productivity shock if the optimal federal RI scheme is in place (red solid lines) and compares these to the impulse responses in autarky (dashed blue lines). Focus first on the case absent federal RI (the dashed blue "autarky" case). Wage rigidity means that the recessionary productivity shock is propagated through the labor market. The job-finding rate falls, the separation rate rises, and workers search less. As a result, the unemployment rate rises sharply and persistently by about 0.4 percentage point. This means that output falls by more than labor productivity alone would suggest. The response of the production tax (lower right panel) illustrates the fiscal mechanics. With labor-market instruments fixed, a rise in separation and a fall in hiring mean fiscal gains to the member state shortly after the shock materializes. To balance the budget, production taxes fall on impact. Eventually, as separation and hiring stabilize, but unemployment remains high, the fiscal cost of unemployment benefits weighs on the budget, so that the production tax needs to rise.

Under the optimal federal RI scheme, the labor-market response is sharply different (see the solid red lines in Figure 2): the separation rate no longer rises in the recession and unemployment fluctuates by an order of magnitude less than in autarky. The key to this is the member state's inherent fiscal response as laid out in Section 2.3. The generous federal transfers mean that the member state receives a sizable, persistent fiscal injection in a recession. By assumption, the labor-market instruments (benefits, layoff taxes, and hiring subsidies) are fixed over the business cycle. In order to balance the government budget, the injection is distributed through cuts in taxes on production, τ_J. The persistent cut in taxes not only raises workers' income, but it also raises the surplus of firms, stimulating hiring and reducing layoffs; all of this stabilizes employment and output, that is, makes the recession in the member state less deep to start with.

Table 5 documents the effect that federal RI has on employment in the current scenario and for the scenarios discussed subsequently. The first column shows the level of average employment in autarky (absent federal RI). The second column shows the effect that the introduction of federal RI has on average employment. The third column shows the effect that federal RI has on the standard deviation of employment. Absent a behavioral response by the member states (first line in the table), federal RI notably reduces (un)employment fluctuations: the standard deviation of employment falls by 86 percent (first row, right-most column). In addition, amid wage rigidity, a fall in the fluctuation of unemployment comes with more stable job-finding rates, which – in the search and matching environment – induces higher average employment in the member state, a mechanism explained in detail in Jung and Kuester (2011). With federal RI, and absent a behavioral response by member states, average employment rises by 1 percent (first row, center column) from an average employment rate of 90.2 percent (left column).

The welfare gains associated with federal RI are documented in Table 6. For all scenarios discussed in the current paper, the table reports the welfare gains associated with the introduction of federal RI. The gains are reported as percentages of equivalent steady-state lifetime consumption. Absent a behavioral response by member states, introducing

Table 5: Federal RI and employment

		Autarky	Effect of federal RI on employment	
		uncond. mean	uncond. mean	standard dev.
Long run	Fixed policies (Section 4.1)	0.902	+1.0%	–86%
	No indexation (Section 4.2.1)			
	b adjusts	0.925	–0.2%	+3%
	τ_ξ and τ_v adjust	0.938	–0.2%	+3%
	Indexation (Section 4.2.2)			
	b adjusts	0.925	–0.5%	–55%
	τ_ξ and τ_v adjust	0.938	–0.9%	–46%
Transition	Indexation (Section 4.3)			
	b adjusts	0.917	–0.6%	–49%
	τ_ξ and τ_v adjust	0.923	–1.1%	–14%

Notes: Table reports unconditional moments for employment both with the federal RI scheme in place and in autarky. Left: long-run mean under autarky (if member states can choose the same set of policies in autarky as after the introduction of federal RI). Center: change in unconditional mean after introducing federal RI and allowing member states to react (+1% meaning that average employment with the federal RI scheme is 1% higher than in autarky). Right: change in the standard deviation of log employment relative to the calibrated baseline (-86% means that after the introduction of federal RI the standard deviation of log employment is 86% lower than in the baseline, where the standard deviation is 2.9; see Table 3).

the optimal federal RI achieves welfare gains that are equivalent to a permanent increase in lifetime consumption of 0.39 percent. For comparison, the welfare costs of business cycles run to 0.4 percent of steady-state consumption in the baseline model, absent RI.

In sum, absent a response by member states, the optimal federal RI would be generous and effective. It would insulate member states against income fluctuations, and it would have the added benefit of reducing distortions that arise from labor-market frictions. The remaining rows of Table 5 and Table 6 suggest, however, that similar gains need not arise in all the scenarios considered below. The following sections explain why.

4.2 Behavioral response by member states, the long-run view

The previous experiment assumed that member states do not adjust their labor-market policies in response to a federal RI scheme. The current section, instead, asks how the shape and scope of federal RI change if member states can adjust labor-market policies. As documented in Section 2.4, in that case the federal RI scheme may have to be shaped so

Table 6: Welfare gains from optimal federal unemployment reinsurance

		consumption equivalent (%)
Long run	Fixed policies (Section 4.1)	0.39
	No indexation (Section 4.2.1)	
	b adjusts	0.0014
	τ_ξ and τ_v adjust	0.0011
	Indexation (Section 4.2.2)	
	b adjusts	0.19
	τ_ξ and τ_v adjust	0.21
Transition	Indexation (Section 4.3)	
	only b adjusts	0.21
	only τ_ξ and τ_v adjust	0.05

Notes: Consumption-equivalent welfare gains from introducing the optimal federal RI scheme (in percent of steady-state consumption; a "1" meaning 1% of steady-state consumption). Welfare gains are computed relative to the same scenario without federal RI. That is, relative to a scenario where member states have the same set of policy options with and without federal RI.

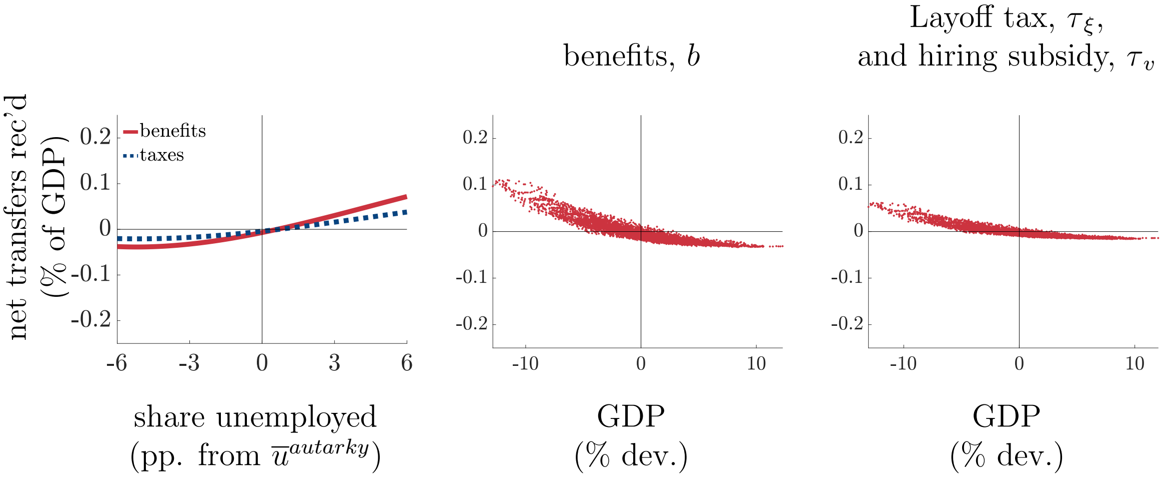
as to account for member states' behavioral responses. We go through each of the two ways alluded to in Section 3.2: the case without indexation and the case with indexation. For now, we continue to focus on the optimal policies that emerge from a long-run perspective.

4.2.1 No indexation: Few transfers

Suppose that payouts under the federal scheme are not indexed to past unemployment rates. Rather, payouts are a function only of current unemployment, $\mathbf{B}_F(u_t^i)$. For this case, Figure 3 presents the shape of the optimal federal RI scheme and the implied federal transfers. The figure presents two different cases. In one ("benefits"), the member state can adjust benefits but not hiring subsidies and layoff taxes. In the other ("taxes"), the member state can adjust the layoff tax and the hiring subsidy, but benefits remain fixed at the level that the member state would choose in autarky. This scenario is meant to highlight that member states' behavioral responses can affect the federal RI scheme even if the level of unemployment benefits would not be under the member states' control, for example, because the unemployment-benefit system is harmonized at the federal level.

The left panel shows the shape of the optimal federal RI scheme. A solid red line marks the case when the member state can adjust benefits only. The blue dots in the left panel

Figure 3: Optimal long-run federal RI scheme – no indexation



Notes: Optimal federal RI without indexation, evaluated using long-run welfare. Left: net transfers received from the rest of the union $\mathbf{B}_F(u_t^i) - \boldsymbol{\tau}_F$ (y-axis), against the share of unemployed workers (x-axis, left panel). The member state can adjust either local UI benefits once and for all (solid red line), or layoff taxes and hiring subsidies (dotted blue line). Center panel: net transfers as percent of GDP (y-axis), against GDP if the member state can adjust benefits; based on 10,000 simulations. Right panel: same as center panel but the member states can adjust only layoff taxes and hiring subsidies.

show the case when the member state can adjust the other labor-market policies only, but not benefits. The remaining two panels simulate the corresponding transfers in the GDP space. Two results emerge. First, accounting for member states' incentives, the schemes have little slope near the steady state (left panel). The constraints imposed on the federal RI scheme by member states' incentives are severe quantitatively. Namely, the payouts are almost two orders of magnitude smaller than absent the member states' response; compare the current Figure 3 to Figure 1. Second, the size of federal transfers tends to be small regardless of which set of instruments the member state is allowed to adjust. Indeed, the member state's behavioral response for taxes constrains the generosity of the federal RI scheme more than a potential change in unemployment benefits. When member states can adjust layoff taxes and hiring subsidies, the federal RI scheme will start paying transfers only if the member state's unemployment rate rises by about two and a half percentage points above the steady-state level (left panel, dotted blue line).[6]

While this shape limits the member state's behavioral response to the federal scheme, as laid out in Section 2.4, it also sharply limits the benefits of federal RI to the member states.

[6]The welfare gains from federal RI are tiny. This makes identifying the optimal shape difficult. The charts shown here constrain the scheme to monotonically increase over the range of unemployment shown. Welfare gains and generosity are similarly small without that constraint.

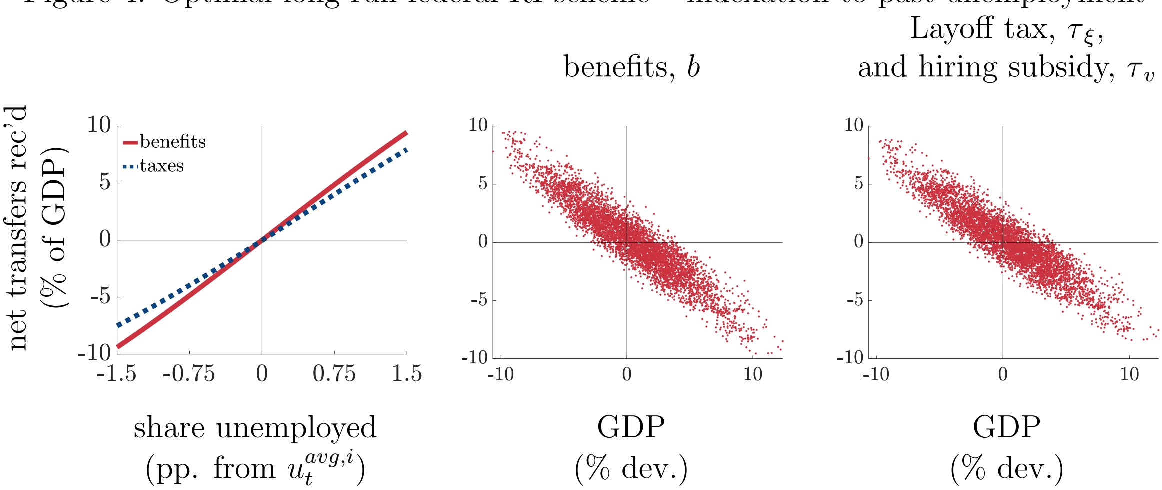
Namely, in line with severely limited generosity, the welfare gains from an introduction of a federal RI scheme are minuscule; see the entry "No indexation (Section 4.2.1)" in Table 6. In addition, federal RI does not provide stabilization gains; see the corresponding entries in Table 5.

4.2.2 Indexation of transfers: Effective federal RI

We have seen that the member states' behavioral responses after federal RI is introduced could severely limit the benefits of federal RI. This raises the question of what element of accountability is effective in mitigating the member states' responses. The current section shows numerically that the simple indexation of payouts to a member state's unemployment history is rather effective, not only in theory (recall Section 2.4) but also quantitatively. In this section, federal transfers take the form $\mathbf{B}_F(u_t^i - u_t^{avg,i})$, with $u_t^{avg,i}$ being the member state's unemployment averaged over the last business cycle. That is, transfers are being paid only when a member state's current unemployment exceeds its "normal" level of unemployment. For now, we keep the focus on the long run.

Figure 4 shows the optimal federal RI scheme and the implied transfers that result for this case. The left panel shows the shape of the federal RI scheme that emerges as optimal.

Figure 4: Optimal long-run federal RI scheme – indexation to past unemployment



Notes: Same as Figure 3, but with indexation of payouts to a member state's past unemployment, $\mathbf{B}_F(u_t^i - u_t^{avg,i})$.

It plots the federal RI scheme against the indicator on which payouts are based. In interpreting the panel, it is important to keep in mind that, relative to the earlier sections, the indicator itself has changed. With indexation to past unemployment, the indicator now is the gap between current unemployment and the average past unemployment, $u_t^i - u_t^{i,avg}$. As before, the solid red line in the left panel shows the shape of the federal RI scheme when the member states can adjust unemployment benefits. The dashed blue line shows the shape of optimal federal RI when the member states can adjust layoff taxes and hiring subsidies. The schemes that emerge are about linear in the indicator. We wish to stress, however, that solving for the optimal federal RI scheme, we did allow for non-linearity, the same as we did in the earlier sections. That the optimal federal RI schemes are about linear, therefore, is a result rather than an assumption.

Due to the change in the indicator, the scale on the x-axis of the left panel naturally differs from that in the left panels of the earlier Figures 1 and 3. The center and right-most panels, instead, are directly comparable to the corresponding panels in those figures. The panels plot net transfers received under the federal RI scheme against a member state's GDP. Quantitatively, the payouts imply sizable redistribution over the business cycle. Payouts align somewhat less well with the business cycle than in Figure 1, however (the clouds are more scattered here, and point toward less than perfect replacement of income).

Table 6 documents that indexation allows for notable welfare gains from introducing federal RI even amid the possibility that member states change their labor-market policies; see the entries under "Indexation (Section 4.2.2)." With indexation, the welfare gains from federal RI run at about two-tenths of a percent of lifetime consumption. The gains are half as large as the gains that prevailed absent a need to account for member states' control over local labor-market policies. The reason is that average unemployment is a slow-moving state variable. Indexation to past unemployment, therefore, makes federal payments less well-timed (the scattered clouds alluded to above).

With indexation, introducing the optimal federal RI scheme makes the standard deviation of employment fall by about half. This is sizable but less than in the fixed-policies baseline; see the entries for "Indexation (Section 4.2.2)" in Table 5. It is important to note that the reduction in business-cycle volatility here does not coincide with a rise in average employment. Rather, introducing federal RI makes average employment *fall*. The reason is that member states change their labor-market policies in a less employment-friendly direction; see Appendix C.

Still, welfare gains from federal RI can arise even if employment falls, precisely because federal RI provides insurance to member states. Absent federal RI, member states choose labor-market policies to partially self-insure against economic fluctuations. In the model at hand, self-insurance is achieved by creating employment relationships. This self-insurance effect can best be seen in the first column of Table 5. The column reports average employment for all the respective scenarios, in autarky. The first row's average employment rate of 0.901 is based on labor-market policies that are optimal for the steady-state economy (as chosen in Section 3.1). For all other scenarios, also in autarky member states can adjust at least one of the labor-market policy instruments. Whenever member states can choose policies in light of business-cycle risk, in autarky they opt for policies that induce higher average employment. They do so in a way that raises the average employment rate by about 2 percent. Generous federal unemployment reinsurance allows member states to reduce these costly efforts to self-insure.

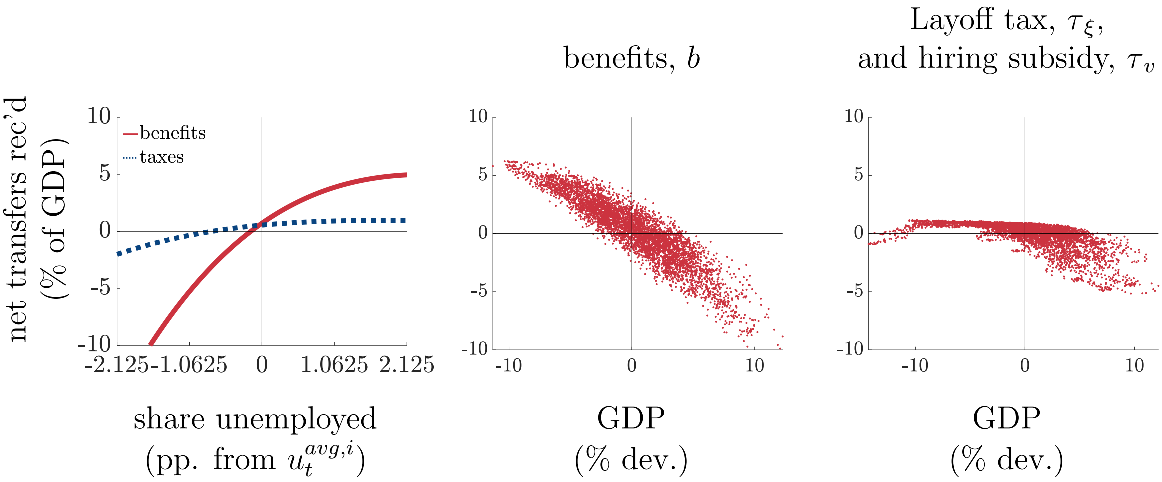
So far, all the calculations have assumed that the economy would immediately jump to the new stochastic steady state. That is, we have focused on the long-run welfare gains. After the introduction of a federal RI scheme, however, it will take time for member-state economies to move to the new non-stochastic steady state. The next section explores how this shapes the effectiveness of indexation.

4.3 Accounting for the transition, with indexation

This section explicitly allows for the transition period. It does so both in evaluating welfare and in searching for the optimal federal RI scheme. We show that the transition period has the potential to affect the scope for federal RI even amid indexation of payouts to past unemployment. The reason for this is simple: on the transition path, indexing payouts to past unemployment may be a weaker deterrent. Past unemployment, by definition, adjusts only gradually to a member state's policy choices. Throughout, we use the calibrated non-stochastic steady state as the initial state for the welfare evaluation.

Figure 5 shows the optimal federal RI schemes that emerge when accounting for the transition period. Clearly, accounting for the transition has a bearing on the optimal federal RI scheme. The schemes that emerge have three features. First, the schemes are less generous at the margin near the steady state than the schemes in Figure 4. The effect is particularly pronounced when member states can adjust layoff taxes and hiring subsidies (the dashed blue line in the left panel). Once more, therefore, it is access to the policy options other than local unemployment benefits that severely limits the scope for federal RI. The reason for this is that access to these (hiring subsidies and layoff taxes) allows member states to exert control over employment effectively, and without having to distort the insurance they provide to workers. This gives member states incentives and the ability

Figure 5: Accounting for the transition, with indexation



Notes: Same as Figure 4, with the difference that – in assessing welfare and in optimizing policies – both the member state and the federal government account for the transition phase. Member states continue to set benefits or layoff taxes and hiring subsidies once and for all in the initial period.

to free-ride during the transition phase. Second, the right tail of the optimal schemes (the tail that provides transfers) is neither linear nor convex. Rather, it is concave. That is, marginal payouts per unemployed worker are higher close to the steady state than at high unemployment rates. For the tax instruments, we find a notably less generous scheme than before. Transfers rise to at most 1 percent of GDP (dashed blue line in the left panel), even in a 10-percent-of-GDP recession (right-most panel). The federal scheme now balances insurance not only against member states' incentives to free-ride in the long run, but also against the incentives to free-ride on the transition path. The way to deter the long-run incentives is through indexation. The way to deter the short-run incentives is by making sure that member states do not set policies that reduce employment too much. The federal RI scheme achieves this by limiting the marginal generosity when moving farther into the tail. The third feature of optimal federal RI when accounting for the transition phase is that the schemes are asymmetric. At the margin, they tax member states in a boom more strongly than they forward transfers to member states in a recession. On net, the schemes thus provide some net transfers also when the unemployment gap is nil. Essentially, the schemes are pre-funded so that the federal RI scheme saves on behalf of the member states. Accounting for the transition barely changes the findings relative to those of Section 4.2.2 – if the member state can only change unemployment benefits. In that case, the welfare gain from federal RI runs at 0.21 percent of lifetime consumption, comparable in magnitude to what the focus on the long run had suggested (see entry "Indexation (Section 4.3)" in Table 6). Also, the effect of federal RI on average employment and its standard deviation is comparable (compare the corresponding entries in Table 5).

Instead, it is the member state's authority over the other two labor-market policies (hiring subsidies and layoff taxes) that restricts the potential generosity of federal RI. Accounting for the transition path, if member states have access to a wider range of labor-market policies, the welfare gain from federal RI falls to 0.05 percent of lifetime consumption, which is a quarter of the gains that emerge with a long-run focus only, and one-eighth of the gains that we found absent a response by member states; see Table 6. Still, there are gains from federal RI. And even this less generous federal RI would notably stabilize the member states' business cycles: federal RI makes the standard deviation of employment fall by 14 percent; compare the last entry of Table 5.

5 Conclusions

What is the scope of a federal unemployment reinsurance scheme (RI) in a union of member states that retain authority over local labor-market policies? The paper has provided theory and a quantitative exploration for a stylized European Monetary Union.

To us, there were two important findings. First, well-designed federal RI remains effective even after accounting for member states' behavioral responses. In particular, indexing federal payouts to past unemployment in the member state in our simulations goes a long way in addressing concerns about free riding. Nevertheless, even such indexation is necessarily imperfect because measured unemployment will only adjust gradually to member states' policy choices. The transition path after the introduction of federal RI may, therefore, limit the generosity that can be provided. Second, in assessing the scope of federal unemployment reinsurance, it matters who has authority over the entire *mix* of labor-market policies. One therefore has to think beyond the local unemployment insurance system. In particular, in our simulations member states' ability to adjust labor-market regulation other than unemployment benefits puts at least as severe a limit on the optimal generosity of federal RI as member states' control over unemployment benefits.

The reader may have several objections to the exercise, all of which are well taken. We focus on a union of atomistically small member states, abstracting from strategic interaction. We do not model externalities across member states or aggregate shocks at the level of the union. We also abstract from heterogeneity *ex-ante* across member states or more pronounced heterogeneity within. And we abstract from labor migration when federal social insurance would bring portability of benefits, which might be conducive to the efficient reallocation of labor. In all of these dimensions, future work seems valuable to us and some of the literature cited in the introduction has started to embark on these endeavors.

In closing, we also wish to add two caveats. First, we have looked at simple federal unemployment reinsurance schemes where payouts to member states depended on unemployment rates or the rise in unemployment relative to a long-run average. We did so for two reasons: because this is the flavor of current policy proposals, and because we looked for federal schemes that might be implementable in the current European institutional setting. We would expect that more complicated schemes (that condition payouts on member states' policy choices or past sequences of shocks, for example), in terms of welfare and stabilization gains, could improve upon the schemes that we have analyzed. Our impression is that the implementation of more complicated schemes in practice might require discretion on the side of some political authority at the community level, meaning a transfer of powers, which, in the past, member states seemed to find it difficult to agree on. But this impression may not reflect reality. Second, we have analyzed federal RI in a setting where business cycles were costly. Part of the cost of fluctuations arose from labor-market distortions that amplified the member-state economies' business cycles. We took these distortions as given. It remains an open question if member states could remove such cyclical distortions in practice and, if so, how these endeavors might interact with federal RI.

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—— Online Appendix ——

A Calculating fourth-order-accurate first moments

In this section we consider a pruned perturbation solution to a dynamic stochastic general equilibrium (DSGE) model. We derive closed-form solutions for fourth-order-accurate unconditional first moments of the model's endogenous variables. The exposition here heavily builds on Andreasen et al. (2018), who already provide the formulas up to third order of accuracy. Our sole contribution is to provide formulas for fourth-order moments. We keep the exposition detailed, repeating several steps in Andreasen et al. (2018), so as to make this appendix self-contained.

A.1 Preliminaries

We consider the following class of DSGE models. Let $y_t \in \mathbb{R}^{n_y}$ be a vector of control variables and $x_t \in \mathbb{R}^{n_x+1}$ a vector of state variables that includes a perturbation parameter $\sigma \geq 0$. Consider a perturbation solution to a DSGE model around the steady state $x_{SS} = 0$. The exact solution to the model is given by

$$\begin{aligned} y_t &= g(x_t), \\ x_{t+1} &= h(x_t) + \sigma\eta\epsilon_{t+1}, \end{aligned} \tag{35}$$

where ϵ_{t+1} follows an n_ϵ dimensional multivariate normal distribution and is independently and identically distributed in each period. Solving a DSGE model amounts to finding unknown functions g and h.

For most DSGE models, the full solution to system (35) cannot be found explicitly. The perturbation solution approximates the true solution using a Taylor series expansion around the steady state, $x_t = x_{t+1} = 0$. Up to fourth order, we have

$$x_{t+1} = h_x x_t + \frac{1}{2} h_{xx} x_t^{\otimes 2} + \frac{1}{6} h_{xxx} x_t^{\otimes 3} + \frac{1}{24} h_{xxxx} x_t^{\otimes 4} + \sigma\eta\varepsilon_{t+1}, \tag{36}$$

where $h_x, h_{xx}, \ldots$ denote first-, second-, etc., order derivatives of function h with respect to vector x. Superscript $^{\otimes n}$ represents the n-th Kronecker power, i.e., $x^{\otimes n} = \overbrace{x \otimes x \otimes \ldots}^{n \text{ times}}$. However, the system (36) may display explosive dynamics and may not have any finite unconditional moments (Andreasen et al., 2018) . The solution to this problem is pruning the state space of the approximated solution so as to remove explosive paths. As shown by Andreasen et al. (2018) the pruned 4th-order approximation to the perturbation solution reads

$$x_{t+1} = x_{t+1}^{f} + x_{t+1}^{s} + x_{t+1}^{rd} + x_{t+1}^{4th},$$

where

$$x_{t+1}^{f} = h_x x_t^{f} + \sigma\eta\varepsilon_{t+1},$$

$$x_{t+1}^{s} = h_x x_t^{s} + \frac{1}{2} h_{xx} \left(x_t^{f}\right)^{\otimes 2},$$

$$x^{rd}_{t+1} = h_x x^{rd}_t + \frac{1}{2} h_{xx} \left(2 \left(\left(x^f_t \otimes x^s_t \right) \right) \right) + \frac{1}{6} h_{xxx} \left(x^f_t \right)^{\otimes 3},$$

and

$$x^{4th}_{t+1} = h_x x^{4th}_t + \frac{1}{2} h_{xx} \left(2 \left(x^f_t \otimes x^{rd}_t \right) + \left(x^s_t \right)^{\otimes 2} \right) + \frac{1}{6} h_{xxx} \left(3 \left(x^f_t \right)^{\otimes 2} \otimes x^s_t \right) + \frac{1}{24} h_{xxxx} \left(x^f \right)^{\otimes 4}.$$

Note that if the shock is drawn from the standard normal distribution, as is the case in the model developed in the current paper, then

$$\begin{aligned} \mathrm{E}\left(\varepsilon_t\right)^{\otimes 2} &= vec(I_{n_e}), \\ \mathrm{E}\left(\varepsilon_t\right)^{\otimes 3} &= 0, \text{and} \\ \mathrm{E}\left(\varepsilon_t\right)^{\otimes 5} &= 0. \end{aligned}$$

Let $M^4 \equiv \mathrm{E}\left(\varepsilon_t\right)^{\otimes 4}$ be the kurtosis of the standard multivariate normal distribution.
In the course of the proofs we will use extensively the following (well-known) properties of the Kronecker product. These are:

$$\begin{aligned} \mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}, \\ (\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} &= \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}, \\ (k\mathbf{A}) \otimes \mathbf{B} &= \mathbf{A} \otimes (k\mathbf{B}) = k(\mathbf{A} \otimes \mathbf{B}), \\ (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} &= \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}), \\ (\mathbf{AC}) \otimes (\mathbf{BD}) &= (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}), \\ vec(ABC) &= (C' \otimes A) vec(B). \end{aligned}$$

We say that matrix $K_{m,n}$ of size $mn \times mn$ is an *commutation matrix* if it has the following property: Let A be an $(m \times n)$ matrix and B a $(p \times q)$ matrix. Then

$$K_{m,p} \left(A \otimes B \right) K_{q,n} = B \otimes A.$$

That is, the commutation matrix reverses the order of Kronecker product. The commutation matrix $K_{m,n}$ can be defined explicitly as

$$K_{n,m} = \sum_{i=1}^{m} \sum_{j=1}^{n} \left((e^m_i (e^n_j)') \otimes (e^n_j (e^m_i)') \right),$$

where e^m_i is the ith unit column vector of order m. For any commutation matrix $K_{p,q} = K^{-1}_{q,p}$.

A.2 Analytical expressions for the first moments

We are ready to derive formulas for unconditional first moments of the endogenous variables. Our goal is to characterize the following expression

$$\mathrm{E}_0\, x_t = \mathrm{E}_0\, x_t^f + \mathrm{E}_0\, x_t^s + \mathrm{E}_0\, x_t^{rd} + \mathrm{E}_0\, x_t^{4th},$$

Andreasen et al. (2018) showed that $\mathrm{E}_0\, x_t^f = \mathrm{E}_0\, x_t^{rd} = 0$. The first equality is the certainty equivalence of the linear approximation. The second equality, $\mathrm{E}_0\, x_t^{rd} = 0$, results from the symmetry of the normal distribution (i.e. skewness is zero).
We write perturbation parameter σ as a separate variable, not included in state x_t. Note that $h_\sigma = h_{x\sigma} = h_{xx\sigma} = h_{xxx\sigma} = h_{xxxx\sigma} = 0$.
For completeness, we derive expressions for the unconditional first moments for the solution approximations of all orders from one up to four. Derivations for orders of approximation up to three are based on Andreasen et al. (2018). Formulas for the fourth order are our contribution.

A.2.1 First- and second-order of approximation

We start with the formulas accurate up to the second order. We have

$$x_{t+1}^f = h_x x_t^f + \sigma\eta\varepsilon_{t+1}$$

$$x_{t+1}^s = h_x x_t^s + \frac{1}{2} h_{xx} \left(x_t^f\right)^{\otimes 2} + \frac{1}{2} h_{\sigma\sigma}\sigma^2$$

$$\begin{aligned}
\mathrm{E}\, x_t^f &= 0 \\
\mathrm{E}\left(x_t^f\right)^{\otimes 2} &= (I_{n_x^2} - h_x \otimes h_x)^{-1}(\sigma^2 \eta \otimes \eta) vec(I)) \\
\mathrm{E}\, x_t^s &= (I_{n_x} - h_x)^{-1}\left[\frac{1}{2} h_{xx}(I_{n_x^2} - h_x \otimes h_x)^{-1}(\sigma^2\eta \otimes \eta) vec(I)) + \frac{1}{2} h_{\sigma\sigma}\sigma^2\right]
\end{aligned}$$

Proof.

$$\begin{aligned}
x_{t+1}^f &= h_x x_t^f + \sigma\eta\epsilon_{t+1} \\
\mathrm{E}\, x_{t+1}^f &= \mathrm{E}\, h_x x_t^f \\
&\text{(stationarity)} \\
\mathrm{E}\, x_t^f (I - h_x) = 0 \;\; &\Rightarrow \;\; \mathrm{E}\, x_t^f = 0.
\end{aligned}$$

$$
\begin{aligned}
\left(x_t^f\right)^{\otimes 2} &= \left(h_x x_t^f + \sigma\eta\epsilon_{t+1}\right) \otimes \left(h_x x_t^f + \sigma\eta\epsilon_{t+1}\right) \\
&= h_x x_t^f \otimes h_x x_t^f + (h_x x_t^f) \otimes (\sigma\eta\epsilon_{t+1}) + (\sigma\eta\epsilon_{t+1}) \otimes (h_x x_t^f) + (\sigma\eta\epsilon_{t+1}) \otimes (\sigma\eta\epsilon_{t+1}) \\
\mathrm{E}\left(x_t^f\right)^{\otimes 2} &= h_x \otimes h_x \,\mathrm{E}\left(x_t^f\right)^{\otimes 2} + \sigma^2\eta \otimes \eta \,\mathrm{E}(\epsilon_{t+1})^{\otimes 2} \\
\mathrm{E}\left(x_t^f\right)^{\otimes 2} &= (I - h_x \otimes h_x)^{-1}(\sigma^2\eta \otimes \eta)vec(I))
\end{aligned}
$$

$$
\begin{aligned}
x_{t+1}^s &= h_x x_t^s + \frac{1}{2}h_{xx}\left(x_t^f\right)^{\otimes 2} + \frac{1}{2}h_{\sigma\sigma}\sigma^2 \\
\mathrm{E}\,x_t^s &= h_x \,\mathrm{E}\,x_t^s + \frac{1}{2}h_{xx}\,\mathrm{E}\left(x_t^f\right)^{\otimes 2} + \frac{1}{2}h_{\sigma\sigma}\sigma^2 \\
\mathrm{E}\,x_t^s &= (I - h_x)^{-1}\left[\frac{1}{2}h_{xx}(I - h_x \otimes h_x)^{-1}(\sigma^2\eta \otimes \eta)vec(I)) + \frac{1}{2}h_{\sigma\sigma}\sigma^2\right].
\end{aligned}
$$

■

A.2.2 Third order

Next, we tackle the third-order approximation. We replicate the results in Andreasen et al. (2018) in the following

$$
x_{t+1}^{rd} = h_x x_t^{rd} + \frac{1}{2}h_{xx}\left(2\left((x_t^f \otimes x_t^s)\right)\right) + \frac{1}{6}h_{xxx}\left(x_t^f\right)^{\otimes 3} + 3 \cdot \frac{1}{6}h_{x\sigma\sigma}x_t^f\sigma^2 + \frac{1}{6}h_{\sigma\sigma\sigma}\sigma^3
$$

$$
\begin{aligned}
\mathrm{E}\,x_{t+1}^f \otimes x_{t+1}^s &= \left(I - h_x^{\otimes 2}\right)^{-1}\left(h_x \otimes \frac{1}{2}h_{xx}\right)\mathrm{E}\left(x_t^f\right)^{\otimes 3} \\
\mathrm{E}\left(x_{t+1}^f\right)^{\otimes 3} &= 0 \\
\mathrm{E}\,x_{t+1}^{rd} &= 0.
\end{aligned}
$$

Proof.

$$\begin{aligned}
x^f_{t+1} \otimes x^s_{t+1} &= \left(h_x x^f_t + \sigma\eta\epsilon_{t+1}\right) \otimes \left(h_x x^s_t + \frac{1}{2} h_{xx} \left(x^f_t\right)^{\otimes 2} + \frac{1}{2} h_{\sigma\sigma}\sigma^2\right) \\
&= (h_x \otimes h_x)\left(x^f_t \otimes x^s_t\right) + \left(h_x \otimes \frac{1}{2} h_{xx}\right)\left(x^f_t\right)^{\otimes 3} + (h_x x_f) \otimes \frac{1}{2} h_{\sigma\sigma}\sigma^2 \\
&\qquad + (\sigma\eta \otimes h_x)\left(\epsilon_{t+1} \otimes x^s_t\right) + (\sigma\eta \otimes .5 h_{xx})\left(\epsilon_{t+1} \otimes \left(x^f_t\right)^{\otimes 2}\right) + (\eta\epsilon_{t+1}) \otimes \frac{1}{2} h_{\sigma\sigma}\sigma^3. \\
\mathrm{E}(x^f_t \otimes x^s_t) &= \left(I - h_x^{\otimes 2}\right)^{-1}\left(h_x \otimes \frac{1}{2} h_{xx}\right)\mathrm{E}\left(x^f_t\right)^{\otimes 3}.
\end{aligned}$$

$$\begin{aligned}
\left(x^f_{t+1}\right)^{\otimes 3} &= \left(x^f_{t+1}\right)^{\otimes 2} \otimes \left(h_x x^f_t + \sigma\eta\epsilon_{t+1}\right) \\
&= \left(h_x x^f_t + \sigma\eta\epsilon_{t+1}\right) \otimes \Bigg[(h_x \otimes h_x)\left(\left(x^f_t\right)^{\otimes 2}\right) + (\sigma\eta \otimes \sigma\eta)\left(\epsilon_{t+1}^{\otimes 2}\right) \\
&\qquad + (\sigma\eta \otimes h_x)\left(\epsilon_{t+1} \otimes x^f_t\right) + (h_x \otimes \sigma\eta)\left(x^f_t \otimes \epsilon_{t+1}\right)\Bigg] \\
&= (h_x \otimes h_x \otimes h_x)\left(x^f_t\right)^{\otimes 3} + (\sigma\eta\epsilon_{t+1}) \otimes (h_x \otimes h_x)\left(\left(x^f_t\right)^{\otimes 2}\right) \\
&\qquad + (h_x x^f_t) \otimes (\sigma\eta \otimes \sigma\eta)\left(\epsilon_{t+1}^{\otimes 2}\right) + (\sigma\eta\epsilon_{t+1}) \otimes (\sigma\eta \otimes \sigma\eta)\left(\epsilon_{t+1}^{\otimes 2}\right) \\
&\qquad + (h_x x^f_t) \otimes (\sigma\eta \otimes h_x)\left(\epsilon_{t+1} \otimes x^f_t\right) + (\sigma\eta\epsilon_{t+1}) \otimes (\sigma\eta \otimes h_x)\left(\epsilon_{t+1} \otimes x^f_t\right) \\
&\qquad + (h_x x^f_t) \otimes (h_x \otimes \sigma\eta)\left(x^f_t \otimes \epsilon_{t+1}\right) + (\sigma\eta\epsilon_{t+1}) \otimes (h_x \otimes \sigma\eta)\left(x^f_t \otimes \epsilon_{t+1}\right) \\
&= (h_x \otimes h_x \otimes h_x)\left(x^f_t\right)^{\otimes 3} + \text{ terms zero in expectation.}
\end{aligned}$$

The last equality follows from the fact that x^f_t and ϵ_{t+1} are independent, therefore $\mathrm{E}\left(x^f_t \otimes \epsilon_{t+1}\right) = \mathrm{E}\, x^f_t \otimes \mathrm{E}\,\epsilon_{t+1} = 0$, $\mathrm{E}(h_x x^f_t) \otimes (\sigma\eta \otimes \sigma\eta)\left(\epsilon_{t+1}^{\otimes 2}\right) = 0$ since $\mathrm{E}(h_x x^f_t) = 0$. Therefore,

$$\mathrm{E}\left(\left(x^f_{t+1}\right)^{\otimes 3}\right) = \mathrm{E}\,(h_x \otimes h_x \otimes h_x)\left(x^f_t\right)^{\otimes 3}$$

and by stationarity $\mathrm{E}\,(h_x \otimes h_x \otimes h_x)\left(x^f_t\right)^{\otimes 3} = 0$,

$$\begin{aligned}
x^{rd}_{t+1} &= h_x x^{rd}_t + \frac{1}{2} h_{xx}\left(2\left((x^f_t \otimes x^s_t)\right)\right) + \frac{1}{6} h_{xxx}\left(x^f_t\right)^{\otimes 3} + 3 \cdot \frac{1}{6} h_{x\sigma\sigma} x^f_t \sigma^2 + \frac{1}{6} h_{\sigma\sigma\sigma}\sigma^3 \\
\mathrm{E}\, x^{rd}_{t+1} &= h_x\, \mathrm{E}\, x^{rd}_t + \frac{1}{6} h_{\sigma\sigma\sigma}\sigma^3
\end{aligned}$$

$$\mathrm{E}\, x^{rd}_t (I - h_x) = 0,$$

the last line following since $h_{\sigma\sigma\sigma} = 0$ for symmetric distributions (see Andreasen et al. 2018). ■

A.2.3 Fourth order

Finally, we derive the solutions accurate up to the fourth order.
We have that the fourth-order accurate law of motion of the states is given by

$$x_{t+1}^{4th} = h_x x_t^{4th} + \frac{1}{2}h_{xx}\left(2\left(x_t^f \otimes x_t^{rd}\right) + (x_t^s)^{\otimes 2}\right) + \frac{1}{6}h_{xxx}\left(3\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right) + \frac{1}{24}h_{xxxx}\left(x^f\right)^{\otimes 4}$$
$$+ \frac{3}{6}h_{\sigma\sigma x}\sigma^2 x_t^s + 6 \cdot \frac{1}{24}h_{\sigma\sigma xx}\sigma^2\left(x_t^f\right)^{\otimes 2} + 4 \cdot \frac{1}{24}h_{\sigma\sigma\sigma x}\sigma^3 x_t^f + \frac{1}{24}h_{\sigma\sigma\sigma\sigma}\sigma^4.$$

The fourth raw moment of the state variables is given by.

$$\mathrm{E}\, x_t^{4th} = (I_{n_x} - h_x)^{-1}\Bigg[\frac{1}{2}h_{xx}\,\mathrm{E}\left(2\left(x_t^f \otimes x_t^{rd}\right) + (x_t^s)^{\otimes 2}\right)$$
$$+ \frac{1}{6}h_{xxx}\,\mathrm{E}\left(3\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right) + \frac{1}{24}h_{xxxx}\,\mathrm{E}\left(x^f\right)^{\otimes 4}$$
$$+ \frac{3}{6}h_{\sigma\sigma x}\sigma^2\,\mathrm{E}\, x_t^s + 6 \cdot \frac{1}{24}h_{\sigma\sigma xx}\sigma^2\,\mathrm{E}\left(x_t^f\right)^{\otimes 2} + \frac{1}{24}h_{\sigma\sigma\sigma\sigma}\sigma^4\Bigg].$$

Where the respective terms for each basis vector of the 4th-order pruned state space are listend in the following.

$$\mathrm{E}\left(x_{t+1}^f\right)^{\otimes 4} = \sigma^2\left(I_{n_x^4} - h_x^{\otimes 4}\right)^{-1}\Bigg[\sigma^2\eta^{\otimes 4}M^4 + \Bigg((h_x^{\otimes 2} \otimes \eta^{\otimes 2})K_{n_e^2,n_x^2}$$
$$+ (h_x \otimes \eta \otimes h_x \otimes \eta)\left(I_{n_x} \otimes K_{n_x,n_e} \otimes I_{n_e}\right)K_{n_e^2,n_x^2}$$
$$+ (h_x \otimes \eta \otimes \eta \otimes h_x)\left(I_{n_x} \otimes K_{n_x,n_e^2}\right)K_{n_e^2,n_x^2}$$
$$+ (\eta \otimes h_x \otimes h_x \otimes \eta)\left(I_{n_e} \otimes K_{n_e,n_x^2}\right)$$
$$+ (\eta \otimes h_x \otimes \eta \otimes h_x)\left(I_{n_e} \otimes K_{n_e,n_x} \otimes I_{n_x}\right)$$
$$+ \left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\Bigg)\left(vec(I_{n_e}) \otimes \mathrm{E}(x_t^f)^{\otimes 2}\right)\Bigg].$$

This we can calculate given $\mathrm{E}\left(x_t^f\right)^{\otimes 2} = \sigma^2(I - h_x^{\otimes 2})^{-1}(\eta^{\otimes 2})vec(I_{n_e}))$ from further above.

Regarding the remaining vectors that span the 4th-order pruned state space, we have

$$\mathrm{E}\left[\left(x_{t+1}^{s}\right)^{\otimes 2}\right] = \left(I_{n_x^2} - h_x^{\otimes 2}\right)^{-1} \Bigg(+ .5\left(K_{n_x,n_x} + I_{n_x^2}\right)\left(h_{xx} \otimes h_x\right) \mathrm{E}\left(\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right) + \frac{1}{4} h_{xx}^{\otimes 2}\,\mathrm{E}(x_t^f)^{\otimes 4}$$
$$+ \left(K_{n_x,n_x} + I_{n_x^2}\right)\left[\frac{1}{2}[(h_x \otimes h_{\sigma\sigma})(\sigma^2 x_t^s)] + \frac{1}{4}(h_{xx} \otimes h_{\sigma\sigma})(\sigma^2 (x_t^f)^{\otimes 2})\right] + \frac{1}{4} h_{\sigma\sigma} \otimes h_{\sigma\sigma}\sigma^4 \Bigg),$$

$$\mathrm{E}\left[\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right] = \left(I_{n_x^3} - h_x^{\otimes 3}\right)^{-1} \Bigg((\sigma^2 \eta^{\otimes 2} \otimes h_x)(\mathrm{E}\left[\epsilon_{t+1}^{\otimes 2}\right] \otimes \mathrm{E}\, x_t^s) + \frac{1}{2}(h_x^{\otimes 2} \otimes h_{xx})\,\mathrm{E}(x_t^f)^{\otimes 4}$$
$$+ \frac{1}{2}(\sigma^2 \eta^{\otimes 2} \otimes h_{xx})(vec(I_{n_e}) \otimes \mathrm{E}(x_t^f)^{\otimes 2}) + \frac{1}{2}(h_x^{\otimes 2} \otimes h_{\sigma\sigma})\sigma^2\,\mathrm{E}\left[(x_t^f)^{\otimes 2}\right]$$
$$+ \frac{1}{2}(\eta^{\otimes 2} \otimes h_{\sigma\sigma})\sigma^4\,\mathrm{E}\,\epsilon_{t+1}^{\otimes 2} \Bigg),$$

$$\mathrm{E}\left[x_{t+1}^f \otimes x_{t+1}^{rd}\right] = \left(I_{n_x^2} - h_x^{\otimes 2}\right)^{-1} \Bigg((h_x \otimes h_{xx})\,\mathrm{E}((x_t^f)^{\otimes 2} \otimes x_t^s) + \frac{1}{6}(h_x \otimes h_{xxx})\,\mathrm{E}\left(x_t^f\right)^{\otimes 4}$$
$$+ \frac{3}{6}(h_x \otimes h_{\sigma\sigma})\sigma^2\,\mathrm{E}\left[(x_t^f)^{\otimes 2}\right] \Bigg).$$

Proof. We derive each of the terms.

$$\begin{aligned}\left(x_{t+1}^f\right)^{\otimes 4} &= \left(h_x x_t^f + \sigma\eta\epsilon_{t+1}\right) \otimes \left(x_{t+1}^f\right)^{\otimes 3} \\ &= (h_x x_t^f) \otimes \left(x_{t+1}^f\right)^{\otimes 3} + (\sigma\eta\epsilon_{t+1}) \otimes \left(x_{t+1}^f\right)^{\otimes 3}. \end{aligned} \tag{37}$$

We tackle separately each of the summands in (37).

$$\begin{aligned}(h_x x_t^f) \otimes \left(x_{t+1}^f\right)^{\otimes 3} &= (h_x x_t^f) \otimes \left(\left[h_x x_t^f + \sigma\eta\epsilon_{t+1}\right] \otimes \left[h_x x_t^f + \sigma\eta\epsilon_{t+1}\right] \otimes \left[h_x x_t^f + \sigma\eta\epsilon_{t+1}\right]\right) \\ &= (h_x x_t^f) \otimes \Bigg(\left[h_x x_t^f + \sigma\eta\epsilon_{t+1}\right] \\ &\quad \otimes \left[h_x x_t^f \otimes h_x x_t^f + h_x x_t^f \otimes \sigma\eta\epsilon_{t+1} + \sigma\eta\epsilon_{t+1} \otimes h_x x_t^f + \sigma\eta\epsilon_{t+1} \otimes \sigma\eta\epsilon_{t+1}\right] \Bigg) \\ &= (h_x x_t^f) \otimes \Bigg(\left[h_x x_t^f + \sigma\eta\epsilon_{t+1}\right] \\ &\quad \otimes \left[h_x^{\otimes 2}(x_t^f)^{\otimes 2} + \sigma(h_x \otimes \eta)(x_t^f \otimes \epsilon_{t+1}) + \sigma(\eta \otimes h_x)(\epsilon_{t+1} \otimes x_t^f) + \sigma^2\eta^{\otimes 2}\epsilon_{t+1}^{\otimes 2}\right] \Bigg). \end{aligned}$$

Further,

$$
\begin{aligned}
(h_x x_t^f) \otimes \left(x_{t+1}^f\right)^{\otimes 3} = (h_x x_t^f) \otimes \Big(& h_x^{\otimes 3}(x_t^f)^{\otimes 3} + \sigma(h_x^{\otimes 2} \otimes \eta)((x_t^f)^2 \otimes \epsilon_{t+1}) \\
& + \sigma(h_x \otimes \eta \otimes h_x)(x_t^f \otimes \epsilon_{t+1} \otimes x_t^f) + \sigma^2(h_x \otimes \eta^{\otimes 2})(x_t^f \otimes \epsilon_{t+1}^{\otimes 2}) \\
& + \sigma(\eta \otimes h_x^{\otimes 2})(\epsilon_{t+1} \otimes (x_t^f)^{\otimes 2}) + \sigma^2(\eta \otimes h_x \otimes \eta)(\epsilon_{t+1} \otimes x_t^f \otimes \epsilon_{t+1}) \\
& + \sigma^2(\eta^{\otimes 2} \otimes h_x)(\epsilon_{t+1}^{\otimes 2} \otimes x_t^f) + \sigma^3 \eta^{\otimes 3} \epsilon_{t+1}^{\otimes 3} \Big).
\end{aligned}
$$

So that

$$
\begin{aligned}
(h_x x_t^f) \otimes \left(x_{t+1}^f\right)^{\otimes 3} = & \, h_x^{\otimes 4}(x_t^f)^{\otimes 4} + \sigma^2(h_x^{\otimes 2} \otimes \eta^{\otimes 2})((x_t^f)^{\otimes 2} \otimes \epsilon_{t+1}^{\otimes 2}) \\
& + \sigma^2 \left(h_x \otimes \eta \otimes h_x \otimes \eta\right) \left(x_t^f \otimes \epsilon_{t+1} \otimes x_t^f \otimes \epsilon_{t+1}\right) \\
& + \sigma^2 \left(h_x \otimes \eta^{\otimes 2} \otimes h_x\right) \left(x_t^f \otimes \epsilon_{t+1}^{\otimes 2} \otimes x_t^f\right) + \text{ terms zero in expectation.}
\end{aligned} \tag{38}
$$

The last equality follows since $\epsilon_{t+1} \perp x_t^f$ and $\mathrm{E}\, x_t^f = \mathrm{E}\left((x_t^f)^{\otimes 3}\right) \mathrm{E}\, \epsilon_{t+1} = 0$.
Using the commutation matrix K_{n_x,n_e} to change the order of $\otimes$. For instance

$$
\begin{aligned}
x_t^f \otimes \epsilon_{t+1} \otimes \epsilon_{t+1} \otimes x_t^f &= x_t^f \otimes I_{n_e}\epsilon_{t+1} \otimes K_{n_x,n_e}(x_t^f \otimes \epsilon_{t+1}) \\
&= x_t^f \otimes (I_{n_e} \otimes K_{n_x,n_e})(\epsilon_{t+1} \otimes x_t^f \otimes \epsilon_{t+1}) \\
&= x_t^f \otimes (I_{n_e} \otimes K_{n_x,n_e})(K_{n_x,n_e}(x_t^f \otimes \epsilon_{t+1}) \otimes \epsilon_{t+1}) \\
&= x_t^f \otimes (I_{n_e} \otimes K_{n_x,n_e})((K_{n_x,n_e} \otimes I_{n_e})(x_t^f \otimes \epsilon_{t+1} \otimes \epsilon_{t+1})) \\
&= [I_{n_x} \otimes (I_{n_e} \otimes K_{n_x,n_e})(K_{n_x,n_e} \otimes I_{n_e})] \left[(x_t^f)^{\otimes 2} \otimes \epsilon_{t+1}^{\otimes 2}\right].
\end{aligned}
$$

This can be simplified further using $(I_r \otimes K_{m,s})(K_{m,s} \otimes I_s) = K_{m,rs}$. Thus

$$
\mathrm{E}\left[x_t^f \otimes \epsilon_{t+1} \otimes \epsilon_{t+1} \otimes x_t^f\right] = \left(I_{n_x} \otimes K_{n_x,n_e^2}\right) \mathrm{E}\left((x_t^f)^{\otimes 2} \otimes \epsilon_{t+1}^{\otimes 2}\right).
$$

Similarly,

$$
\mathrm{E}\left(x_t^f \otimes \epsilon_{t+1} \otimes x_t^f \otimes \epsilon_{t+1}\right) = (I_{n_x} \otimes K_{n_x,n_e} \otimes I_{n_e})\, \mathrm{E}\left((x_t^f)^{\otimes 2} \otimes \epsilon_{t+1}^{\otimes 2}\right).
$$

Going back to (38) we have

$$
\begin{aligned}
\mathrm{E}\left[(h_x x_t^f) \otimes \left(x_{t+1}^f\right)^{\otimes 3}\right] = & \, h_x^{\otimes 4}\, \mathrm{E}(x_t^f)^{\otimes 4} + \sigma^2 \Big[(h_x^{\otimes 2} \otimes \eta^{\otimes 2}) \\
& + \left(h_x \otimes \eta \otimes h_x \otimes \eta\right) \left(I_{n_x} \otimes K_{n_x,n_e} \otimes I_{n_e}\right) \\
& + \left(h_x \otimes \eta \otimes \eta \otimes h_x\right) \left(I_{n_x} \otimes K_{n_x,n_e^2}\right) \Big] \left(\mathrm{E}\left[(x_t^f)^{\otimes 2}\right] \otimes vec(I_{n_e})\right),
\end{aligned}
$$

since $E\left((x_t^f)^{\otimes 2}\otimes\epsilon_{t+1}^{\otimes 2}\right)=\mathrm{E}\left[(x_t^f)^{\otimes 2}\right]\otimes vec(I_{n_e})$.

Regarding the other summand in (37),

$$
\begin{aligned}
(\sigma\eta\epsilon_{t+1})\otimes\left(x_{t+1}^f\right)^{\otimes 3} &= (\sigma\eta\epsilon_{t+1})\otimes\Big(h_x^{\otimes 3}(x_t^f)^{\otimes 3}+\sigma(h_x^{\otimes 2}\otimes\eta)((x_t^f)^2\otimes\epsilon_{t+1}) \\
&\quad +\sigma(h_x\otimes\eta\otimes h_x)(x_t^f\otimes\epsilon_{t+1}\otimes x_t^f)+\sigma^2(h_x\otimes\eta^{\otimes 2})(x_t^f\otimes\epsilon_{t+1}^{\otimes 2}) \\
&\quad +\sigma(\eta\otimes h_x^{\otimes 2})(\epsilon_{t+1}\otimes(x_t^f)^{\otimes 2})+\sigma\eta\epsilon_{t+1}\otimes\sigma(h_x\otimes\eta)(x_t^f\otimes\epsilon_{t+1}) \\
&\quad +\sigma\eta\epsilon_{t+1}\otimes\sigma(\eta\otimes h_x)(\epsilon_{t+1}\otimes x_t^f)+\sigma^3\eta^{\otimes 3}\epsilon_{t+1}^{\otimes 3}\Big) \\
&= \sigma^2\left(\eta\otimes h_x\otimes h_x\otimes\eta\right)\left(\epsilon_{t+1}\otimes x_t^f\otimes x_t^f\otimes\epsilon_{t+1}\right) \\
&\quad +\sigma^2\left(\eta\otimes h_x\otimes\eta\otimes h_x\right)\left(\epsilon_{t+1}\otimes x_t^f\otimes\epsilon_{t+1}\otimes x_t^f\right) \\
&\quad +\sigma^2\left(\eta^{\otimes 2}\otimes h_x^{\otimes 2}\right)\left(\epsilon_{t+1}^{\otimes 2}\otimes(x_t^f)^{\otimes 2}\right)+\sigma^4\eta^{\otimes 4}\epsilon_{t+1}^{\otimes 4} \\
&\quad +\text{ terms that are zero in expectation.}
\end{aligned}
\tag{39}
$$

Using the commutation matrices

$$
\begin{aligned}
\mathrm{E}\left(\epsilon_{t+1}^{\otimes 2}\otimes(x_t^f)^{\otimes 2}\right) &= vec(I_{n_e})\otimes\mathrm{E}(x_t^f)^{\otimes 2} \\
\mathrm{E}\left(\epsilon_{t+1}\otimes x_t^f\otimes\epsilon_{t+1}\otimes x_t^f\right) &= (I_{n_e}\otimes K_{n_e,n_x}\otimes I_{n_x})\left(vec(I_{n_e})\otimes\mathrm{E}(x_t^f)^{\otimes 2}\right) \\
\mathrm{E}\left(\epsilon_{t+1}\otimes x_t^f\otimes x_t^f\otimes\epsilon_{t+1}\right) &= (I_{n_e}\otimes K_{n_e,n_x^2})(vec(I_{n_e})\otimes\mathrm{E}(x_t^f)^{\otimes 2}).
\end{aligned}
$$

Plugging into (39) delivers

$$
\begin{aligned}
\mathrm{E}\left[(\sigma\eta\epsilon_{t+1})\otimes\left(x_{t+1}^f\right)^{\otimes 3}\right] &= \sigma^2\Big[\left(\eta\otimes h_x\otimes h_x\otimes\eta\right)\left(I_{n_e}\otimes K_{n_e,n_x^2}\right) \\
&\quad +\left(\eta\otimes h_x\otimes\eta\otimes h_x\right)\left(I_{n_e}\otimes K_{n_e,n_x}\otimes I_{n_x}\right) \\
&\quad +\left(\eta^{\otimes 2}\otimes h_x^{\otimes 2}\right)\Big]\left(vec(I_{n_e})\otimes\mathrm{E}(x_t^f)^{\otimes 2}\right)+\sigma^4\eta^{\otimes 4}M^4,
\end{aligned}
$$

where $M^4\equiv\mathrm{E}\left[\epsilon_{t+1}^{\otimes 4}\right]$.

Going back to the original formula (37),

$$
\begin{aligned}
\mathrm{E}\left(x_{t+1}^f\right)^{\otimes 4} &= \mathrm{E}\left[(h_x x_t^f)\otimes\left(x_{t+1}^f\right)^{\otimes 3}\right] + \mathrm{E}\left[(\sigma\eta\epsilon_{t+1})\otimes\left(x_{t+1}^f\right)^{\otimes 3}\right]\\
&= h_x^{\otimes 4}\,\mathrm{E}(x_t^f)^{\otimes 4} + \sigma^2\Big[(h_x^{\otimes 2}\otimes\eta^{\otimes 2})\\
&\quad + (h_x\otimes\eta\otimes h_x\otimes\eta)\left(I_{n_x}\otimes K_{n_x,n_e}\otimes I_{n_e}\right)\\
&\quad + (h_x\otimes\eta\otimes\eta\otimes h_x)\left(I_{n_x}\otimes K_{n_x,n_e^2}\right)\Big]\left(\mathrm{E}\left[(x_t^f)^{\otimes 2}\right]\otimes vec(I_{n_e})\right)\\
&+\sigma^2\Big[(\eta\otimes h_x\otimes h_x\otimes\eta)\left(I_{n_e}\otimes K_{n_e,n_x^2}\right)\\
&\quad + (\eta\otimes h_x\otimes\eta\otimes h_x)\left(I_{n_e}\otimes K_{n_e,n_x}\otimes I_{n_x}\right)\\
&\quad + \left(\eta^{\otimes 2}\otimes h_x^{\otimes 2}\right)\Big]\left(vec(I_{n_e})\otimes \mathrm{E}(x_t^f)^{\otimes 2}\right) + \sigma^4\eta^{\otimes 4}M^4.
\end{aligned}
$$

Hence, using stationarity and the fact that $\mathrm{E}\left[(x_t^f)^{\otimes 2}\right]\otimes vec(I_{n_e}) = K_{n_e^2,n_x^2}\left(vec(I_{n_e})\otimes\mathrm{E}(x_t^f)^{\otimes 2}\right)$, we have that

$$
\begin{aligned}
\mathrm{E}\left(x_{t+1}^f\right)^{\otimes 4} = \sigma^2\left(I_{n_x^4}-h_x^{\otimes 4}\right)^{-1}\Big[&\sigma^2\eta^{\otimes 4}M^4 + \Big((h_x^{\otimes 2}\otimes\eta^{\otimes 2})K_{n_e^2,n_x^2}\\
&+ (h_x\otimes\eta\otimes h_x\otimes\eta)\left(I_{n_x}\otimes K_{n_x,n_e}\otimes I_{n_e}\right)K_{n_e^2,n_x^2}\\
&+ (h_x\otimes\eta\otimes\eta\otimes h_x)\left(I_{n_x}\otimes K_{n_x,n_e^2}\right)K_{n_e^2,n_x^2}\\
&+ (\eta\otimes h_x\otimes h_x\otimes\eta)\left(I_{n_e}\otimes K_{n_e,n_x^2}\right)\\
&+ (\eta\otimes h_x\otimes\eta\otimes h_x)\left(I_{n_e}\otimes K_{n_e,n_x}\otimes I_{n_x}\right)\\
&+ \left(\eta^{\otimes 2}\otimes h_x^{\otimes 2}\right)\Big)\left(vec(I_{n_e})\otimes\mathrm{E}(x_t^f)^{\otimes 2}\right)\Big].
\end{aligned}
$$

Continued on next page.

Next, we calculate $\mathrm{E}\left(\left(x_{t+1}^s\right)^{\otimes 2}\right)$.

$$\begin{aligned}
\left(x_{t+1}^s\right)^{\otimes 2} &= \left(h_x x_t^s + \frac{1}{2}h_{xx}\left(x_t^f\right)^{\otimes 2} + \frac{1}{2}h_{\sigma\sigma}\sigma^2\right) \otimes \left(h_x x_t^s + \frac{1}{2}h_{xx}\left(x_t^f\right)^{\otimes 2} + \frac{1}{2}h_{\sigma\sigma}\sigma^2\right) \\
&= h_x^{\otimes 2}(x_t^s)^{\otimes 2} + (h_x x_t^s) \otimes (\frac{1}{2}h_{xx}\left(x_t^f\right)^{\otimes 2}) + (\frac{1}{2}h_{xx}\left(x_t^f\right)^{\otimes 2}) \otimes (h_x x_t^s) \\
&\quad + \left(\frac{1}{2}h_{xx}\left(x_t^f\right)^{\otimes 2}\right) \otimes \left(\frac{1}{2}h_{xx}\left(x_t^f\right)^{\otimes 2}\right) \\
&\quad + \frac{1}{2}(h_{\sigma\sigma} \otimes h_x)(\sigma^2 x_t^s) + \frac{1}{4}(h_{\sigma\sigma} \otimes h_{xx})(\sigma^2 (x_t^f)^{\otimes 2}) + \frac{1}{4}h_{\sigma\sigma} \otimes h_{\sigma\sigma}\sigma^4 \\
&\quad + \frac{1}{2}(h_x \otimes h_{\sigma\sigma})(\sigma^2 x_t^s) + \frac{1}{4}(h_{xx} \otimes h_{\sigma\sigma})(\sigma^2 (x_t^f)^{\otimes 2}) \\
&= h_x^{\otimes 2}(x_t^s)^{\otimes 2} + (\frac{1}{2}h_x \otimes h_{xx})\left(x_t^s \otimes \left(x_t^f\right)^{\otimes 2}\right) + \left(\frac{1}{2}h_{xx} \otimes h_x\right)\left(\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right) \\
&\quad + \frac{1}{4}h_{xx}^{\otimes 2}(x_t^f)^{\otimes 4} + \frac{1}{2}\left(h_x \otimes h_{\sigma\sigma} + h_{\sigma\sigma} \otimes h_x\right)(\sigma^2 x_t^s) \\
&\quad + \frac{1}{4}\left(h_{\sigma\sigma} \otimes h_{xx} + h_{xx} \otimes h_{\sigma\sigma}\right)(\sigma^2 (x_t^f)^{\otimes 2}) + \frac{1}{4}h_{\sigma\sigma} \otimes h_{\sigma\sigma}\sigma^4 \\
&= h_x^{\otimes 2}(x_t^s)^{\otimes 2} + (\frac{1}{2}h_x \otimes h_{xx})K_{n_x^2,n_x}\left(\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right) + \left(\frac{1}{2}h_{xx} \otimes h_x\right)\left(\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right) \\
&\quad + \frac{1}{4}h_{xx}^{\otimes 2}(x_t^f)^{\otimes 4} + \frac{1}{2}\left(K_{n_x,n_x} + I_{n_x^2}\right)[(h_x \otimes h_{\sigma\sigma})(\sigma^2 x_t^s)] \\
&\quad + \frac{1}{4}\left(K_{n_x,n_x} + I_{n_x^2}\right)([h_{xx} \otimes h_{\sigma\sigma})(\sigma^2 (x_t^f)^{\otimes 2})] + \frac{1}{4}h_{\sigma\sigma} \otimes h_{\sigma\sigma}\sigma^4 \\
&= h_x^{\otimes 2}(x_t^s)^{\otimes 2} + \frac{1}{2}(h_{xx} \otimes h_x)\left(K_{n_x^2,n_x} + I_{n_x^2}\right)\left(\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right) + \frac{1}{4}h_{xx}^{\otimes 2}(x_t^f)^{\otimes 4} \\
&\quad + \left(K_{n_x,n_x} + I_{n_x^2}\right)\left[\frac{1}{2}[(h_x \otimes h_{\sigma\sigma})(\sigma^2 x_t^s)] + \frac{1}{4}(h_{xx} \otimes h_{\sigma\sigma})(\sigma^2 (x_t^f)^{\otimes 2})\right] + \frac{1}{4}h_{\sigma\sigma} \otimes h_{\sigma\sigma}\sigma^4.
\end{aligned}$$

where the last line follows since $(h_x \otimes h_{xx})K_{n_x^2,n_x} = K_{n_x,n_x}(h_{xx} \otimes h_x)$ (Note: $K_{p,q}^{-1} = K_{q,p}$, so $K_{n_x,n_x} = K_{n_x,n_x}^{-1}$ and $K_{1,n} = K_{n,1} = I_n$).
Hence

$$\begin{aligned}
\mathrm{E}\left[\left(x_{t+1}^s\right)^{\otimes 2}\right] = \left(I_{n_x^2} - h_x^{\otimes 2}\right)^{-1}\Bigg(&+ .5\left(K_{n_x,n_x} + I_{n_x^2}\right)(h_{xx} \otimes h_x)\,\mathrm{E}\left(\left(x_t^f\right)^{\otimes 2} \otimes x_t^s\right) + \frac{1}{4}h_{xx}^{\otimes 2}\,\mathrm{E}(x_t^f)^{\otimes 4} \\
&+ \left(K_{n_x,n_x} + I_{n_x^2}\right)\left[\frac{1}{2}[(h_x \otimes h_{\sigma\sigma})(\sigma^2 x_t^s)] + \frac{1}{4}(h_{xx} \otimes h_{\sigma\sigma})(\sigma^2 (x_t^f)^{\otimes 2})\right] + \frac{1}{4}h_{\sigma\sigma} \otimes h_{\sigma\sigma}\sigma^4\Bigg).
\end{aligned}$$

Continued on next page.

Next we calculate $\mathrm{E}\left(\left(x_{t+1}^f\right)^{\otimes 2}\otimes x_{t+1}^s\right)$.

$$\begin{aligned}\left(x_{t+1}^f\right)^{\otimes 2}\otimes x_{t+1}^s &= \left(h_x x_t^f\otimes h_x x_t^f + (h_x x_t^f)\otimes(\sigma\eta\epsilon_{t+1}) + (\sigma\eta\epsilon_{t+1})\otimes(h_x x_t^f) + (\sigma\eta\epsilon_{t+1})\otimes(\sigma\eta\epsilon_{t+1})\right)\\ &\quad\otimes\left(h_x x_t^s + \frac{1}{2}h_{xx}\left(x_t^f\right)^{\otimes 2} + \frac{1}{2}h_{\sigma\sigma}\sigma^2\right)\\ &= h_x^{\otimes 3}((x_t^f)^{\otimes 2}\otimes x_t^s) + (\sigma^2\eta^{\otimes 2}\otimes h_x)(\epsilon_{t+1}^{\otimes 2}\otimes x_t^s) + \frac{1}{2}(h_x^{\otimes 2}\otimes h_{\sigma\sigma})\sigma^2(x_t^f)^{\otimes 2}\\ &\quad + \frac{1}{2}(\eta^{\otimes 2}\otimes h_{\sigma\sigma})\sigma^4\epsilon_{t+1}^{\otimes 2} + \frac{1}{2}(h_x^{\otimes 2}\otimes h_{xx})(x_t^f)^{\otimes 4} + \frac{1}{2}(\sigma^2\eta^{\otimes 2}\otimes h_{xx})(\epsilon_{t+1}^{\otimes 2}\otimes(x_t^f)^{\otimes 2})\\ &\quad + \text{ terms zero in expectation.}\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{E}\left[\left(x_t^f\right)^{\otimes 2}\otimes x_t^s\right] &= \left(I_{n_x^3} - h_x^{\otimes 3}\right)^{-1}\Bigg((\sigma^2\eta^{\otimes 2}\otimes h_x)(\mathrm{E}\left[(\epsilon_{t+1})^{\otimes 2}\right]\otimes\mathrm{E}\, x_t^s) + \frac{1}{2}(h_x^{\otimes 2}\otimes h_{xx})\,\mathrm{E}(x_t^f)^{\otimes 4}\\ &\quad + \frac{1}{2}(\sigma^2\eta^{\otimes 2}\otimes h_{xx})(vec(I_{n_e})\otimes\mathrm{E}(x_t^f)^{\otimes 2}) + \frac{1}{2}(h_x^{\otimes 2}\otimes h_{\sigma\sigma})\sigma^2\,\mathrm{E}\left[(x_t^f)^{\otimes 2}\right]\\ &\quad + \frac{1}{2}(\eta^{\otimes 2}\otimes h_{\sigma\sigma})\sigma^4\,\mathrm{E}\,\epsilon_{t+1}^{\otimes 2}\Bigg),\end{aligned}$$

where the equality follows since ϵ_{t+1} and x_t^s are orthogonal.

For the last missing terms.

$$\begin{aligned}x_{t+1}^f\otimes x_{t+1}^{rd} &= \left(h_x x_t^f + \sigma\eta\varepsilon_{t+1}\right)\otimes\Bigg(h_x x_t^{rd} + \frac{1}{2}h_{xx}\left(2\left((x_t^f\otimes x_t^s)\right)\right) + \frac{1}{6}h_{xxx}\left(x_t^f\right)^{\otimes 3}\\ &\quad + \frac{3}{6}h_{\sigma\sigma x}\sigma^2 x_t^f + \frac{1}{6}h_{\sigma\sigma\sigma}\sigma^3\Bigg)\\ &= h_x^{\otimes 2}(x_{t+1}^f\otimes x_{t+1}^{rd}) + (h_x\otimes h_{xx})((x_t^f)^{\otimes 2}\otimes x_t^s) + \frac{1}{6}(h_x\otimes h_{xxx})\left(x_t^f\right)^{\otimes 4}\\ &\quad + \frac{3}{6}(h_x\otimes h_{\sigma\sigma})\sigma^2(x_t^f)^{\otimes 2} + \text{ terms zero in expectation}\end{aligned}$$

so that

$$\begin{aligned}\mathrm{E}\left[x_{t+1}^f\otimes x_{t+1}^{rd}\right] &= \left(I_{n_x^2} - h_x^{\otimes 2}\right)^{-1}\Bigg((h_x\otimes h_{xx})\,\mathrm{E}((x_t^f)^{\otimes 2}\otimes x_t^s) + \frac{1}{6}(h_x\otimes h_{xxx})\,\mathrm{E}\left(x_t^f\right)^{\otimes 4}\\ &\quad + \frac{3}{6}(h_x\otimes h_{\sigma\sigma})\sigma^2\,\mathrm{E}\left[(x_t^f)^{\otimes 2}\right]\Bigg)\end{aligned}$$

This completes the proof. ■

A.2.4 Simplifying the 4th-order expressions

For the model used in the current paper, the 4th-order Kronecker products can be simplified further. Namely,

$$\left(h_x^{\otimes 2} \otimes \eta^{\otimes 2}\right) K_{n_e^2, n_x^2} = K_{n_x^2, n_x^2} \left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)$$

For the current model we have $n_e = 1$ so that $K_{n_e,n_x} = K_{n_x,n_e} = I_{n_x}$, $I_{n_e} = vec(I_{n_e}) = 1$.

$$\begin{aligned}
\mathrm{E}\left(x_{t+1}^f\right)^{\otimes 4} &= \sigma^2 \left(I_{n_x^4} - h_x^{\otimes 4}\right)^{-1} \Big[\sigma^2 \eta^{\otimes 4} M^4 + K_{n_x^2,n_x^2}\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right) \\
&\quad + \left(h_x \otimes \eta \otimes h_x \otimes \eta\right)\left(I_{n_x} \otimes I_{n_x}\right) \\
&\quad + \left(h_x \otimes \eta \otimes \eta \otimes h_x\right)\left(I_{n_x} \otimes I_{n_x}\right) \\
&\quad + \left(\eta \otimes h_x \otimes h_x \otimes \eta\right) \\
&\quad + \left(\eta \otimes h_x \otimes \eta \otimes h_x\right)\left(I_{n_x} \otimes I_{n_x}\right) \\
&\quad + \left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\Big)\left(\mathrm{E}(x_t^f)^{\otimes 2}\right)\Big] \\
&= \sigma^2 \left(I_{n_x^4} - h_x^{\otimes 4}\right)^{-1} \Big[\sigma^2 \eta^{\otimes 4} M^4 \\
&\quad + \left(h_x \otimes \eta \otimes h_x \otimes \eta\right)\left(I_{n_x} \otimes I_{n_x}\right) \\
&\quad + \left(h_x \otimes \eta \otimes \eta \otimes h_x\right)\left(I_{n_x} \otimes I_{n_x}\right) \\
&\quad + \left(\eta \otimes h_x \otimes h_x \otimes \eta\right) \\
&\quad + \left(\eta \otimes h_x \otimes \eta \otimes h_x\right)\left(I_{n_x} \otimes I_{n_x}\right) \\
&\quad + \left(K_{n_x^2,n_x^2} + I_{n_x^2}\right)\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\Big)\left(\mathrm{E}(x_t^f)^{\otimes 2}\right)\Big].
\end{aligned}$$

Note that $I_{n_x} \otimes I_{n_x} = I_{n_x^2}$ so that

$$\begin{aligned}
\mathrm{E}\left(x_{t+1}^f\right)^{\otimes 4} &= \sigma^2 \left(I_{n_x^4} - h_x^{\otimes 4}\right)^{-1} \Big[\sigma^2 \eta^{\otimes 4} M^4 \\
&\quad + \left(h_x \otimes \eta \otimes h_x \otimes \eta\right) \\
&\quad + \left(h_x \otimes \eta \otimes \eta \otimes h_x\right) \\
&\quad + \left(\eta \otimes h_x \otimes h_x \otimes \eta\right) \\
&\quad + \left(\eta \otimes h_x \otimes \eta \otimes h_x\right) \\
&\quad + \left(K_{n_x^2,n_x^2} + I_{n_x^2}\right)\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\Big)\left(\mathrm{E}(x_t^f)^{\otimes 2}\right)\Big].
\end{aligned}$$

Moreover

$$\begin{aligned}
h_x \otimes \eta \otimes h_x \otimes \eta &= h_x \otimes \left[K_{n_x,n_x}\left(h_x \otimes \eta\right) K_{n_e,n_x}\right] \otimes \eta \\
&= \left(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x}\right)\left(h_x^{\otimes 2} \otimes \eta^{\otimes 2}\right) \\
&= \left(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x}\right)\left(K_{n_x^2,n_x^2}\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\right),
\end{aligned}$$

$$
\begin{aligned}
h_x \otimes \eta \otimes \eta \otimes h_x &= h_x \otimes \eta \otimes \left[K_{n_x,n_x}\left(h_x \otimes \eta\right) K_{n_e,n_x}\right] \\
&= \left(I_{n_x^2} \otimes K_{n_x,n_x}\right)\left[h_x \otimes \eta \otimes h_x \otimes \eta\right] \\
&= \left(I_{n_x^2} \otimes K_{n_x,n_x}\right)\left[\left(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x}\right)\left(h_x^{\otimes 2} \otimes \eta^{\otimes 2}\right)\right] \\
&= \left(I_{n_x^2} \otimes K_{n_x,n_x}\right)\left[\left(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x}\right)\left(K_{n_x^2,n_x^2}\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right),\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\eta \otimes h_x \otimes \eta \otimes h_x &= \eta \otimes \left[K_{n_x,n_x}\left(\eta \otimes h_x\right) K_{n_x,n_e}\right] \otimes h_x \\
&= (I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x})(\eta^{\otimes 2} \otimes h_x^{\otimes 2}),
\end{aligned}
$$

$$
\begin{aligned}
\eta \otimes h_x \otimes h_x \otimes \eta &= \eta \otimes h_x \otimes \left[K_{n_x,n_x}(\eta \otimes h_x) K_{n_x,n_e}\right] \\
&= (I_{n_x^2} \otimes K_{n_x,n_x})(\eta \otimes h_x \otimes \eta \otimes h_x) \\
&= (I_{n_x^2} \otimes K_{n_x,n_x})\left[(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x})(\eta^{\otimes 2} \otimes h_x^{\otimes 2})\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathrm{E}\left(x_{t+1}^f\right)^{\otimes 4} = \sigma^2 \left(I_{n_x^4} - h_x^{\otimes 4}\right)^{-1} \Bigg[& \sigma^2 \eta^{\otimes 4} M^4 \\
& + (I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x})\left(K_{n_x^2,n_x^2}\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\right) \\
& + \left(I_{n_x^2} \otimes K_{n_x,n_x}\right)\left[(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x})\left(K_{n_x^2,n_x^2}\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\right)\right] \\
& + (I_{n_x^2} \otimes K_{n_x,n_x})\left[(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x})(\eta^{\otimes 2} \otimes h_x^{\otimes 2})\right] \\
& + (I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x})(\eta^{\otimes 2} \otimes h_x^{\otimes 2}) \\
& + \left(K_{n_x^2,n_x^2} + I_{n_x^2}\right)\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right) \Bigg) \left(\mathrm{E}(x_t^f)^{\otimes 2}\right) \Bigg],
\end{aligned}
$$

or

$$
\begin{aligned}
\mathrm{E}\left(x_{t+1}^f\right)^{\otimes 4} = \sigma^2 \left(I_{n_x^4} - h_x^{\otimes 4}\right)^{-1} \Bigg[& \sigma^2 \eta^{\otimes 4} M^4 \\
& + \Bigg\{ (I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x}) K_{n_x^2,n_x^2} \\
& + \left(I_{n_x^2} \otimes K_{n_x,n_x}\right)\left[(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x})\left(K_{n_x^2,n_x^2}\right)\right] \\
& + (I_{n_x^2} \otimes K_{n_x,n_x})\left[(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x})\right] \\
& + (I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x}) \\
& + \left(K_{n_x^2,n_x^2} + I_{n_x^4}\right) \Bigg\} \left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right) \Bigg) \left(\mathrm{E}(x_t^f)^{\otimes 2}\right) \Bigg],
\end{aligned}
$$

$$\begin{aligned}
\mathrm{E}\left(x_{t+1}^{f}\right)^{\otimes 4} = \sigma^2 \left(I_{n_x^4} - h_x^{\otimes 4}\right)^{-1} \Bigg[& \sigma^2 \eta^{\otimes 4} M^4 \\
& + \Bigg\{ \left(I_{n_x^2} \otimes K_{n_x,n_x}\right)\left(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x}\right) + I_{n_x^4} \\
& + I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x} \Bigg\} \left(K_{n_x^2,n_x^2} + I_{n_x^4}\right)\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\left(\mathrm{E}(x_t^f)^{\otimes 2}\right) \Bigg],
\end{aligned}$$

hence

$$\begin{aligned}
\mathrm{E}\left(x_{t+1}^{f}\right)^{\otimes 4} = \sigma^2 \left(I_{n_x^4} - h_x^{\otimes 4}\right)^{-1} \Bigg[& \sigma^2 \eta^{\otimes 4} M^4 \\
& + \Bigg\{ \left(I_{n_x^2} \otimes K_{n_x,n_x} + I_{n_x^4}\right)\left(I_{n_x} \otimes K_{n_x,n_x} \otimes I_{n_x}\right) + I_{n_x^4} \Bigg\} \left(K_{n_x^2,n_x^2} + I_{n_x^4}\right)\left(\eta^{\otimes 2} \otimes h_x^{\otimes 2}\right)\left(\mathrm{E}(x_t^f)^{\otimes 2}\right) \Bigg].
\end{aligned}$$

B Finding optimal policies and welfare gains

This section describes the algorithm that we use to derive the optimal federal RI scheme. As discussed in Section 3.2, we wish to have closed-form expressions for the value of the objective function (for given member-state policies and a given federal RI scheme). To obtain optimal federal RI in Sections 4.1 and 4.2, we find federal and member-state policies that maximize the unconditional mean of the objective function. When accounting for the transition in Section 4.3 we find policies that maximize the conditional expectation of the objective. In these cases, we condition on the initial state being the non-stochastic steady state implied by our calibrated model (Table 4).

The next section describes how we find the optimal federal RI scheme. We describe this for the case when the member state can choose policies once and we account for the transitions. The other cases are handled analogously.

B.1 Finding the optimal federal RI scheme

The algorithm proceeds as follows.

1. Fix Chebyshev nodes for the federal RI scheme as described in Section 3.2. Keep nodes fixed.

2. The goal is to find values $\boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3, \phi_4]' \in \mathbb{R}$ and $\boldsymbol{\tau}_F$ that solve the federal government's problem (24) anticipating the member states' policy choices. The values of $\boldsymbol{\phi}$ induce payout function $\mathbf{B}_F(\,\cdot\,;\,\cdot\,)$.

3. Find $\boldsymbol{\phi}$ by numerical optimization. For this, for each try $\boldsymbol{\phi}$, evaluate the federal government's objective function using either unconditional expectations (Sections 4.1 to 4.2) or conditional expectations for a given initial state (Section 4.3).

4. In order to evaluate the federal planner's objective function for given $\boldsymbol{\phi}$, make sure that the scheme is feasible in light of the member states' optimal response. In particular, a federal RI policy has to be self-financing in light of member states' responses; recall (22). For fixed ϕ we iterate as follows.

   (a) mark the iteration by $^{(n)}$. Set $n = 0$. Start from an initial value of $\boldsymbol{\tau}_F^{(-1)}$.

   (b) set $\boldsymbol{\tau}_F = \boldsymbol{\tau}_F^{(n-1)}$. For given federal RI policy $\boldsymbol{\phi}$ and $\boldsymbol{\tau}_F = \boldsymbol{\tau}_F^{(n-1)}$, let the member state solve (26).

   (c) label the maximizing member-state policies $\{\tau_v^{i,(n)}, \tau_\xi^{i,(n)}, b^{i,(n)}\}$. These induce a law of motion $\mu_0^{(n)}$ and a value for the objective function of $\int W_0^{(n)} d\mu_0^{(n)}$.

   (d) for given member-state policies $\{\tau_v^{i,(n)}, \tau_\xi^{i,(n)}, b^{i,(n)}\}$, and given federal policy $\boldsymbol{\phi}$, find a value $\boldsymbol{\tau}_F^{(n)}$ that solves the federal RI scheme's financing constraint (22) for these given policies and given the induced dynamics for the member-state economies.

   (e) if $\boldsymbol{\tau}_F^{(n)}$ is not sufficiently close to $\boldsymbol{\tau}_F^{(n-1)}$, set $n = n + 1$ and go to step 4b. Else, set $\boldsymbol{\tau}_F = \boldsymbol{\tau}_F^{(n)}$ and go to step 5.

5. The federal policy implied by $\boldsymbol{\phi}$ and $\boldsymbol{\tau}_F$ is feasible. Set $\int W_0 d\mu_0 = \int W_0^{(n)} d\mu_0^{(n)}$.

6. Continue numerical optimization started in step 3 until the maximum for the federal government's objective is found.

C Values of the labor-market instruments in each case

Table 7: Instrument values for each scenario, and implied steady state and fluctuation

		instruments			steady state		
		b	τ_ξ	τ_v	$\frac{\bar{c}_u}{\bar{c}_e}$	$\bar{e}$	std(e)
Long run	Section 4.1						
	autarky	0.379	6.390	0.796	0.518	0.909	2.90
	federal RI	0.379	6.390	0.796	0.518	0.910	0.42
	Section 4.2.1, no indexation						
	b adjusts						
	autarky	0.367	6.390	0.796	0.503	0.927	2.03
	federal RI	0.382	6.390	0.796	0.505	0.925	2.10
	τ_ξ *and* τ_v *adjust*						
	autarky	0.379	7.215	0.813	0.520	0.942	1.81
	federal RI	0.379	7.151	0.811	0.520	0.940	1.87
	Section 4.2.2, indexation						
	b adjusts						
	autarky	0.367	6.390	0.796	0.503	0.927	2.03
	federal RI	0.373	6.390	0.796	0.511	0.919	0.92
	τ_ξ *and* τ_v *adjust*						
	autarky	0.379	7.215	0.813	0.520	0.942	1.81
	federal RI	0.379	6.862	0.797	0.520	0.929	0.98
Transition	Section 4.3, indexation						
	b adjusts						
	autarky	0.372	6.390	0.796	0.509	0.921	2.31
	federal RI	0.378	6.390	0.796	0.515	0.915	1.17
	τ_ξ *and* τ_v *adjust*						
	autarky	0.379	6.806	0.803	0.520	0.927	2.28
	federal RI	0.379	6.508	0.829	0.515	0.920	1.97

Notes: For each of the scenarios reported in the main text, the table reports the value of the labor-market instruments (first three columns). The table also reports information for the implied steady state, namely, the effective replacement rate of consumption c_u/c_e and employment (next two columns). Last the table reports the value of the standard deviation of log employment (in percent, last column). For each scenario the table reports the case with federal RI and absent federal RI (autarky).