Panel Income Changes and Changing Relative Income Inequality*

Robert Duval-Hernández[1], Gary S. Fields[†2] and George H. Jakubson[3]

[1]Open University of Cyprus, UCY, CIDE, and IZA
[2]Cornell University, IZA, and WIDER
[3]Cornell University

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Abstract

When economic growth (or economic decline) takes place, who benefits and who is hurt, and by how much? The more traditional way of answering this question is to compare two or more comparable cross sections and gauge changing income inequality among individuals, households, or countries. A newer way is to utilize longitudinal data and assess the pattern of panel income changes. How do these two approaches relate to one another? Using various inequality measures, we establish that for each such measure, there is a corresponding panel income-change regression such that whenever inequality falls, there will be convergent panel income changes; thus, for that specific panel income-change regression, it is impossible to have divergence along with falling cross-sectional inequality. However, if we leave the choice of inequality measure/income change regression

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[†]**Corresponding Author:** gsf2@cornell.edu

pair unrestricted, then any combination of rising/falling inequality and convergent/divergent panel income changes can arise. Finally, we derive precise conditions underlying the above four possible combinations.
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1 Introduction

Who benefits and who is hurt how much in income terms when an economy grows or contracts?[1] The more traditional way of answering this question is to compare data from two or more anonymous cross sections and gauge changing income inequality among individuals or households. Calculations of cross-sectional inequality measures such as Gini coefficients, income shares of particular quantiles of the income distribution, and comparisons of Lorenz curves have a long and distinguished history. A more recent technique within the anonymous tradition is to calculate Growth Incidence Curves (GICs) which, by design, compare the growth of incomes among anonymous quantiles of the income distribution (Ravallion and Chen, 2003).

A newer way of gauging who benefits and who is hurt is to utilize data on a panel of people and assess the pattern of panel income changes, allowing people to change quantiles. Often called income mobility analysis, the assessment of panel income changes usually is carried out by means of regressions capturing income dynamics (e.g. Atkinson *et al.*, 1992), or by constructing what are called mobility profiles (e.g. Grimm, 2007; Van Kerm, 2009) or, synonymously, non-anonymous Growth Incidence Curves (Bourguignon, 2011).

The fundamental difference between the cross-sectional data and panel data approaches is this. When working with comparable cross-sections and looking at income inequality using such familiar tools as Lorenz curves and inequality indices, the analyst looks at the income of whoever is in the p'th position in each distribution (initial and final) regardless of whether that

[1] Throughout this paper, "income" will be used as shorthand for whichever magnitude is under examination.

is the same person in one distribution as in the other.[2] By contrast, when looking at panel income changes, the analyst first identifies which individual is in the p'th position in the initial distribution and follows that person over time, even if that person is in a different position later on.

Thus, a statement about the persons in a particular group g, say, the richest 1% or poorest 10%, means different things in the two approaches. The standard inequality analysis permits statements of the type "the *anonymous* richest 1% got richer while the *anonymous* poorest 10% got poorer" while the panel data analysis makes a different type of statement: "those who *started* in the richest 1% experienced income changes of such and such amount while those who *started* in the poorest 10% experienced income changes of a different amount." To the extent that people move around within the income distribution, the two approaches provide different information.

In the literature, the anonymous approach has been explored in much more detail than the panel one. However, as Bourguignon (2011) argues in the context of income growth of countries in the world distribution of mean incomes: "*if one is interested in whether global growth has been pro-poor* [...] *there does not seem to be any good reason for ignoring what happened to countries that grew fast enough to move out of the bottom deciles*" [emphasis in the original].

In practice, both approaches are meaningful. On the one hand, the panel approach allows us to compare the gains and losses of identified individuals over time, and thus it lets us explore whether initially disadvantaged individuals catch-up with the rest of the population. However, for many applications the information provided by the anonymous approach is sufficient. For instance, in the debate on the political economy implications of growing inequality (Stiglitz, 2013, 2015; Bourguignon, 2015), the gap between the top and bottom of the income distribution is more relevant than the income-origin of the rich and the powerful.

Numerous empirical studies have shown that the exact same data can produce markedly different patterns depending on whether the anonymous or panel approach is used. For analyses comparing the two approaches applied to the income growth of the same individuals or households, see Dragoset and Fields (2008) on the United States, Grimm (2007) on Indonesia and Peru, Khor and Pencavel (2010) on China, Palmisano and Peragine (2015) on Italy, Duval-Hernández *et al.* (2017) on Mexico, Fields *et al.* (2015) on Argentina,

[2] The inequality literature terms this "anonymity".

Mexico, and Venezuela, and Jenkins and Van Kerm (2016) on Britain, among others. See also Bourguignon (2011) on growth of mean incomes for countries in different deciles of the world per capita income distribution.

Are cross-sectional changes favoring the anonymous rich over the anonymous poor necessarily accompanied by panel income changes favoring the panel rich over the panel poor, and likewise for the anonymous poor and panel poor?

The idea that a pattern of panel changes whereby those initially at the bottom gain more than those initially at the top necessarily results in falling inequality was first raised by Francis Galton in 1886 in the context of the distribution of heights among parents and children. Later scholars demonstrated that no such implication holds, and Galton's assertion has come to be dubbed "Galton's fallacy" (see, for example, Bliss, 1999).

The literature also offers a claim regarding the opposite set of circumstances. Consider a panel of countries with per capita incomes in comparable currency units - Purchasing Power Parity-adjusted dollars, for example. Define β-divergence (convergence) as arising when a regression of final log-income on initial log-income produces a regression coefficient greater than (less than) one. Define σ-convergence (divergence) as arising when the variance of log-incomes falls (rises) from the initial year to the final year. It is proven in the literature that β-divergence measured in this way and σ-convergence measured in this way cannot arise simultaneously - more specifically, σ-convergence implies β-convergence, but β-convergence does not imply σ-convergence (Furceri, 2005; Wodon and Yitzhaki, 2006).

Is it possible to have convergent panel income changes- that is, the income changes we see following panel individuals over time tend to decrease as initial income is increased- and simultaneously to have rising income inequality? Is it possible to have divergent panel income changes along with falling income inequality? Are the possibilities in times of economic growth different from those in times of economic decline? Under what conditions do these different possibilities arise?

Contrary to the suggestions in the preceding paragraphs, it is easy to show that it is indeed possible to have rising or falling inequality along with convergent or divergent panel income changes. To illustrate this it suffices to work with a simple two-person example. For instance, convergent panel changes can coexist with rising inequality when the dollar incomes in this fictitious economy evolve as in the vectors below,

$$[1, 3] \to [5, 1].$$

In this example the initially poor individual experiences a gain of 4 dollars, while the initially rich one experiences a loss of 2 dollars. Clearly, panel income changes are convergent, while at the same time inequality increased.

Now consider a different economy with the following transition

$$[5, 20] \to [7, 23].$$

In this case the gains of the initially rich are larger than those of the initially poor (3 dollars versus 2 dollars, respectively), therefore panel dollar changes are divergent. Yet, the income share of the poor grew from 20% to 23%, so in this two-person economy, relative inequality fell.

The above examples measured panel income changes in dollar terms. However, we can generate transitions illustrating that all combinations of rising/falling relative inequality can coexist with various measures of convergence and divergence in panel income changes, both in times of economic growth and in times of economic decline.[3]

The purpose of this paper is to derive precise theoretical conditions reconciling various measures of rising/falling inequality together with various measures of convergent/divergent panel income changes; and identifying when certain impossibilities arise.

Our paper is not the first one to derive such conditions. In addition to the aforementioned contributions by Furceri (2005) and Wodon and Yitzhaki (2006), Jenkins and Van Kerm (2006) decompose changes in Generalized Gini indices into a term reflecting share convergence and a term reflecting re-ranking. Similarly, Nissanov and Silber (2009) propose an alternative reconciliation of β- and σ-convergence, as defined above.

Our contribution to this literature is that unlike the studies just cited, our reconciliation of changes in inequality and panel income changes is made using very general and widely used measures of both phenomena. In particular, our analysis of inequality changes is made first through commonly used inequality indices like the coefficient of variation, the variance of log-incomes, the Gini, the Atkinson, and the Generalized Entropy indices. Then,

[3] A comprehensive list of examples is presented in Section S.6 in a Supplementary file that accompanies this paper.

we provide results for cases of Lorenz-curve dominance.[4] Similarly, for the analysis of panel income changes, we rely on the analysis of linear regressions between initial and final incomes, as traditionally used in studies of intra and inter-generational income mobility (e.g. Atkinson *et al.*, 1992; Solon, 1999, respectively), and the macro literature on absolute convergence of mean per capita incomes across countries (e.g. Barro, 1991; Sala-i-Martin, 1996).[5] By offering a reconciliation of widely used measures of inequality and panel income changes, we thus provide a framework that can be used by these several literatures.

Overall, the results derived in this paper indicate that for each of the aforementioned inequality measures it is possible to find a corresponding income-change regression such that, whenever inequality falls, there will necessarily be convergent panel income changes of a specified type. Intuitively, these results mean that whenever the anonymous distribution of income becomes less dispersed, the incomes of the initially poor and the initially rich converge to one another.

However, this intuitive conclusion crucially hinges on the matching of particular measures of inequality with specific income-change regressions. More precisely, if we leave unrestricted the choice of inequality measure/income-change regression pair, it is possible to find income processes where inequality falls, yet where there is divergence in panel income changes (as in the example listed above).

Our paper goes beyond the previous literature in deriving precise conditions under which the four possibilities of rising/falling inequality can coexist with convergent/divergent panel income changes, and in identifying the pairings of inequality measures and panel income-change regressions for which falling inequality implies convergence in panel incomes. These conditions are derived in Section 3 and summarized in Section 4. First, we define our terms precisely.

[4]Furthermore, in cases when the Lorenz curves cross, we additionally analyze in the Supplementary file, changes in inequality using the family of transfer-sensitive inequality indices, whenever one distribution third-order stochastically dominates another one.

[5]In the macroeconomics literature the term "absolute convergence" is used when the only explanatory variable in the regression is initial income.

2 Concepts and Definitions

The two key variables in this research are income inequality and panel income changes. “Income” is the term used for the economic variable of interest, which could be total income, labor earnings, consumption, or something else. The income recipient will be called a “person”, but the results apply equally to households, workers, per capitas, adult equivalents, or country means.

2.1 Notation

Consider an economy with n individuals observed over two time periods, initial (or 0), and final (or 1).[6]

Denote by d_{it} the income of individual i in period t measured in constant monetary units (e.g., real dollars). We drop the individual subindex i to denote vectors, e.g., $d_t = (d_{1t}, d_{2t}, \ldots, d_{nt})'$.

The basic building block of panel data analysis is the panel data matrix $\mathbf{D} = [d_0, d_1]$. If each column of $\mathbf{D}$ is divided by its respective mean, μ_t, we obtain the resulting matrix of shares $\mathbf{S} = [s_0, s_1]$.[7]

In addition to income shares, we will also deal with other strictly monotonic transformations of income, like log-incomes, denoted by $\log d_t$. More generically, we will denote by $y_t = f(d_t)$ a variable of income in dollars transformed by a specified strictly monotonically increasing function $f(\cdot)$.

A crucial feature of the panel data matrix $\mathbf{D}$ is that it involves pairs of incomes for each individual, which implies that if the i-th element of d_0 is moved to another row, the i-th element of d_1 must also be moved to the same row. In other words, in panel data analyses we are allowed to permute entire rows of $\mathbf{D}$, a property we call *Multi-period Anonymity.* This contrasts with the property of *Single-period Anonymity* (or simply *Anonymity*) commonly used in the analysis of cross-sectional inequality. Under single-period anonymity, we are allowed to separately permute a given column of $\mathbf{D}$ without necessarily permuting the elements in other columns of the data matrix. In mobility studies then, the assumption of single-period anonymity is replaced by multi-

[6] Our analysis in this paper is limited to income changes between an initial period and a final period.

[7] The inequality literature usually works with income expressed as a share of total income. In order to make an easier link with the regressions involving share changes, we will work throughout with shares of mean income. It is obvious that inequality comparisons are the same for shares of total income as for shares of mean income.

period anonymity, where the income *trajectories* matter without having to look at the names of the particular individuals experiencing such trajectories.

For the most part, income vectors and their transformations are sorted in ascending order of individuals' *initial-period* incomes.[8] An exception to this is when the final incomes are sorted in ascending order of *final-period* income; the resulting vector will be used in the Lorenz curve calculations below.

Definition 1 ***Vector of Final Shares in Ascending Order.***
Let $P(\cdot)$ be a permutation operator. Then, define $s_c = (s_{1c}, \ldots, s_{nc})$ ("c" for "counterfactual") as the final income-share vector when final incomes are sorted in ascending order of final *income, i.e.*

$$s_c \equiv P(s_1) \quad \text{such that} \quad s_{ic} \leq s_{jc} \ \forall\, i \leq j. \tag{1}$$

It is useful to illustrate the relationship between $\mathbf{D}$, $\mathbf{S}$, and $\mathbf{s_c}$ with a simple example. In particular, we display next a particular panel data matrix $\mathbf{D}$, together with its corresponding matrices $\mathbf{S}$, and $\mathbf{s_c}$.

$$\mathbf{D} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 10 & 9 \end{bmatrix}; \ \mathbf{S} = \begin{bmatrix} 0.23 & 0.69 \\ 0.46 & 0.23 \\ 2.31 & 2.08 \end{bmatrix}; \ \mathbf{s_c} = \begin{bmatrix} 0.23 \\ 0.69 \\ 2.08 \end{bmatrix}$$

Let r_{it} denote the population-normalized rank of individual i in period t, when the distribution in period t is sorted in ascending order of income in *that same period*. In other words, if R_{it} is the rank of individual i when the distribution is sorted in ascending order of income in period t, the normalized rank equals $r_{it} = R_{it}/n$.

Throughout this paper, the notation $[y_{10}, y_{20}, \ldots, y_{n0}] \to [y_{11}, y_{21}, \ldots, y_{n1}]$ will be used to denote the change in an income variable y for panel people $1, 2, \ldots, n$ from time 0 to time 1.

Another concept that we will need is that of a Rank-Preserving Transfer, defined next.

Definition 2 ***Equalizing Rank-Preserving Transfer***
A rank-preserving equalizing transfer $h > 0$ is a transfer of income between

[8] This sorting is required for some inequality measures and immaterial for the convergence regressions.

two individuals with ranks i and j and with dollar incomes $d_{j0} > d_{i0}$, such that:

$$\begin{aligned}
&d_{k0} = d_{k1} && \text{for } k \neq i, j, \\
&d_{j1} = d_{j0} - h, && \\
&d_{i1} = d_{i0} + h, && \text{where:} \\
&\text{if } j = i + 1, && h < (d_{j0} - d_{i0})/2; \\
&\text{if } j > i + 1, && h < \min[(d_{i+1,0} - d_{i0}), (d_{j0} - d_{j-1,0})].
\end{aligned}$$

A rank-preserving disequalizing transfer is defined similarly.[9] Equalizing transfers are sometimes called "progressive transfers", while disequalizing transfers are sometimes called "regressive transfers".

With this notation we can now define how we will measure inequality and convergence/divergence of panel income changes.

2.2 Income Inequality

When is income inequality rising or falling? Income inequality and the change in income inequality are conceptualized and measured in a number of ways. "Relative inequality" is concerned with income comparisons measured in terms of ratios, "absolute inequality" with income comparisons measured in terms of dollar differences. In this paper we focus on relative inequality exclusively.

The way we measure inequality change in this paper is completely standard (e.g. Sen, 1997; Foster and Sen, 1997; Cowell, 2011), namely, we use the Lorenz functional (defined next) or a suitable inequality index to represent the inequality at two points in time and then to compare them.

Definition 3 ***Lorenz Curve***
Let s_{jt} be the income-share of the individual in position j in period t, when shares are sorted in ascending order of income in that period. The Lorenz Curve of income in period t, LC_t, is the continuous piecewise linear function connecting the points

$$(F_i, L_i) = (i/n, \sum_{j=1}^{i} s_{jt}/n)$$

[9] In this case, the final income of the poorer individual will be $d_{i1} = d_{i0} - h$, the final income of the richer individual will be $d_{j1} = d_{j0} + h$, and the last two conditions are replaced by $h < \min[d_{i,0} - d_{i-1,0}, d_{j+1,0} - d_{j,0}]$.

where $(F_0, L_0) = (0, 0)$.

A powerful and widely-used criterion for determining which of two income distributions is relatively more equal than another is the three-part Lorenz criterion, which states i) if Lorenz curve A lies somewhere above and never below Lorenz curve B, A is more equal than B, ii) if Lorenz curves A and B coincide, then A and B are equally unequal, and iii) if the Lorenz curves of A and B cross, the relative inequalities of A and B cannot be compared using the Lorenz criterion alone. To formally express the above criterion we need the following definition.

Definition 4 ***Lorenz Dominance***
Let s_{j0} *be the initial income-share of the individual in position j, when shares are sorted in ascending order of initial income. Let* s_{jc} *be the final income-share of the individual in position j, when shares are sorted in ascending order of final income. The final income distribution Lorenz-dominates the initial distribution whenever*

$$\begin{aligned} &s_{1c} + s_{2c} + \ldots + s_{jc} \geq s_{10} + s_{20} + \ldots + s_{j0} \quad \text{for } j = 1, 2, \ldots, n-1 \text{ and} \\ &s_{1c} + s_{2c} + \ldots + s_{jc} > s_{10} + s_{20} + \ldots + s_{j0} \quad \text{for some } j < n. \end{aligned} \tag{2}$$

Following standard notation, $LC_1 \succ LC_0$ means that the Lorenz curve in period 1 dominates that of period 0, namely, incomes in period 1 are more equally distributed than the ones in period 0 according to the Lorenz-criterion. This situation is sometimes also referred as a "Lorenz-improvement" when going from d_0 to d_1. The judgement is commonly made that when the final income distribution is more equal than the initial one, LC_1 is preferred to LC_0 by the Lorenz criterion. If the domination is weak we denote it as $LC_1 \succeq LC_0$, which means that incomes in period 1 are at least as equally distributed as those in period 0 by the Lorenz criterion. Similarly, if the previous inequalities are reversed we talk of a "Lorenz-worsening".

Judging a Lorenz-dominant distribution to be more equal than a Lorenz-dominated one is equivalent to making inequality comparisons on the basis of four commonly-accepted relative inequality axioms: anonymity, scale-independence, population-independence, and the transfer principle (Fields and Fei, 1978).

Yet, despite its appeal, the Lorenz criterion is not universally used for two reasons: it is ordinal, and it is incomplete. When the Lorenz criterion does render a verdict about which of two income distributions is more equal than another, it can only say that distribution A is more equal than distribution B but not how much more equal A is than B. And when Lorenz curves cross, the Lorenz criterion cannot render a verdict.

Those analysts who seek a complete cardinal comparison of the inequalities of two income distributions are led to use one or more inequality indices, $I(\cdot)$. For present purposes, these indices can be put into three categories:

1. Lorenz-consistent relative inequality indices: An inequality index is Lorenz-consistent if, when one Lorenz curve dominates another, the index registers the dominant distribution as (strictly) more equal (strong Lorenz-consistency) or equally unequal (weak Lorenz-consistency). A partial listing of strongly Lorenz-consistent relative inequality indices includes the Gini coefficient, Atkinson index, Theil index, the Generalized Entropy family of indices, and the coefficient of variation. Included among the weakly Lorenz-consistent inequality indices are the income share of the richest X%, income share of the poorest Y%, and the decile ratios (e.g. 90-10). For details, see Sen (1997), Foster and Sen (1997), and Cowell (2011).

2. Lorenz-inconsistent relative inequality indices: An inequality index is Lorenz-inconsistent if, when one Lorenz curve dominates another, it is ever the case that the index shows the Lorenz-dominant distribution to be less equal. One commonly-used relative inequality index is Lorenz-inconsistent: the variance of the logarithms of income. This index violates the transfer principle - that is, it is possible to make a rank-preserving transfer of income from a relatively rich person to a relative poorer person and yet the index can register an increase in relative inequality (Foster and Ok, 1999; Cowell, 2011).

3. Transfer-sensitive inequality indices: These indices are Lorenz-consistent, but in addition they can also unanimously rank distributions even in the presence of crossings in Lorenz-curves, as long as one distribution third-order stochastically dominates another. All members of the Atkinson index family, the Theil index, and more generally all of the Generalized Entropy measures with parameter smaller than 2 are "transfer-sensitive". The Gini index, however is not (Shorrocks and Foster, 1987).

In our work below, we emphasize Lorenz curve comparisons and Lorenz-consistent inequality indices. However, we give attention to the variance of log-incomes despite its Lorenz-inconsistency, because of its widespread use in the literature.

2.3 Divergent and Convergent Panel Income Changes

By definition, income mobility analysis entails looking at the joint distribution of individuals' incomes at two or more points in time. This entails an analysis of panel income changes since we follow particular individuals over time.

The income mobility literature distinguishes six mobility concepts: time-independence, positional movement, share movement, directional income movement, non-directional income movement, and mobility as an equalizer of longer-term incomes relative to initial (Fields, 2008). For purposes of characterizing the pattern of panel income changes in this paper, the relevant concept is directional income movement among panel people - that is, who gains or loses how much, from an initial date to a final one.

Panel income changes are said to be divergent when the income recipients who started ahead on average get ahead faster than those who started behind. It is convergent when those who started ahead on average get ahead more slowly (or fall behind more) than those who started behind. It is neutral when neither is the case.

What it means to get ahead at a faster, slower, or same rate itself requires careful specification. In the macroeconomics literature, the object of interest is nearly always the growth rate in percentages, often approximated by changes in log-income (see, for example, Barro, 1991; Sala-i-Martin, 1996). On the other hand, the literature on panel income changes among individuals or households presents a more varied picture; some studies use income changes in dollars, while others use changes in log-dollars, exact percentage changes, changes in income shares, or changes in income quantiles such as deciles or centiles (see Jäntti and Jenkins, 2015, for a recent and comprehensive review of the literature).

Much of the literature assesses divergence or convergence by assuming a linear relationship between final income and initial income or between income change and initial income. In this paper, we follow this approach as well. Accordingly, we gauge divergence or convergence as follows.

Definition 5 ***Convergence and Divergence***
For a generic income variable y, which can be measured in dollars or by

a strictly monotonically increasing function of dollars $y = f(d)$, define the final-on-initial regression

$$y_1 = \alpha_y + \beta_y y_0 + u_y \tag{3.1}$$

and the change-on-initial regression

$$\Delta y \equiv y_1 - y_0 = \gamma_y + \delta_y y_0 + u_y. \tag{3.2}$$

The two regressions are linked by the relationship $\delta_y = \beta_y - 1$. Divergence of panel changes in y arises when $\beta_y > 1$, or equivalently, when $\delta_y > 0$, and convergence when $\beta_y < 1$, or equivalently, when $\delta_y < 0$. Otherwise, the panel changes in y are deemed neutral.

An alternative way of estimating convergence in shares is through the share-change on initial-y regressions

$$\Delta s = \kappa_y + \lambda_y y_0 + e_y, \tag{4}$$

in which case there will be "share-change-on-y" convergence whenever $\lambda_y < 0$, divergence if $\lambda_y > 0$; otherwise the share changes are deemed neutral. To emphasize, different regressions (and thus λ_y parameters) will arise depending on the specific transformation $y_0 = f(d_0)$ selected.

Finally, define the regression of the exact proportional changes in dollars on initial dollars

$$\text{pch d} \equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}. \tag{5}$$

Divergence of exact proportional changes arises when $\theta > 0$, convergence when $\theta < 0$. Otherwise, the exact proportional change patterns are deemed neutral.

Since relative inequality analysis is concerned with the distribution of income shares, it is natural to compare it to a regression also expressed in shares. Hence, the usefulness of conducting analysis using regressions (3.1) and (3.2) for $y = s = d/\mu$, as well as with equation (4). In this case, we subscript the parameters of the generic regressions (3.1) and (3.2) with the letter "s".

In spite of this natural connection between relative inequality and a share-change regression, often when someone is interested in finding out whether "the rich got richer and the poor, poorer" the reference is to changes in dollars and not merely in shares. For this reason we will also study changes in dollars. In this case, we subscript the parameters of the generic regressions (3.1) and (3.2) with "d".

Finally, in many applications, economists have been interested in studying whether proportional income changes are convergent or divergent. In particular they have studied whether on average initially richer individuals had proportional income changes larger than those of initially poorer individuals.

We can approximate proportional changes using a log-log regression or we can measure them exactly. In the first case we will use generic equations (3.1) and (3.2) for the transformation $y = \log d$, and we subscript their parameters with "log". For the analysis of exact proportional panel changes we use equation (5).

3 Mathematical Results

In this section we analytically develop a set of results that establish the connection between changes in relative inequality and our several income change concepts.

In what follows we will derive our results under two maintained assumptions. First, we will assume that in the initial period the income distribution is not completely equal. That is, we assume $V(y_0) > 0$. If by contrast, we were to allow cases where $V(y_0) = 0$, all the slope coefficients in our regressions would be undefined. The second assumption we will maintain is that between the initial and final periods, there is a change in relative inequality. In Section 3.5 we will briefly discuss the case when inequality remains unchanged.

In everything that follows we consider regressions done on population and abstract from all issues of inference. The proofs of all the results are included in the Supplementary file accompanying this paper.

3.1 Inequality Measures and Panel Changes in Dollars

Many analysts are especially concerned with the relationship between panel changes in dollars and changes in relative inequality. The study of panel dollar changes is relevant because when economic growth takes place, it might be too easy to find convergent proportional gains, for the simple reason that the poor start from much lower income levels. Percentage changes can converge even if the dollar gains of the initially poor individuals are smaller than those of the initially rich ones, and so studying the patterns of dollar changes rather than percentage changes in the panel provides a stronger test of whether incomes are converging or diverging when the economy is growing.

We can derive a connection between dollar regressions, either in final-on-initial form

$$d_1 = \alpha_d + \beta_d d_0 + u_d \tag{6}$$

or in change-on-initial form

$$\Delta d = \gamma_d + \delta_d d_0 + u_d \tag{7}$$

and the coefficient of variation. Unlike the variance of dollars (which is not scale-independent), the coefficient of variation has the advantages of being scale-independent and Lorenz-consistent.

Proposition 1 ***Changes in the Coefficient of Variation, Convergence in Dollars, and Economic Growth***

Let β_d *be defined by the final-on-initial dollar regression (6), and denote the correlation coefficient from this regression by* ρ_d*. Let* $CV(d_t)$ *denote the coefficient of variation of income at period* t*, and let* g *denote the economy-wide growth rate in incomes between year 0 and year 1. Then there is divergence/convergence in dollars as follows:*

$$\beta_d \gtreqless 1 \qquad (\text{i.e. } \delta_d \gtreqless 0) \iff \rho_d \frac{CV(d_1)}{CV(d_0)}(1+g) \gtreqless 1. \tag{8}$$

A look at equation (8) shows that in order to make a rising coefficient of variation compatible with convergent dollar changes ($\beta_d < 1$), we must either have a sufficiently strong economic decline ($g < 0$) or a sufficiently low intertemporal correlation ρ_d.

Consider an economy in which economic growth has taken place (i.e., $g > 0$) and income inequality as measured by the coefficient of variation has risen ($CV(d_1) > CV(d_0)$). If initial and final incomes were perfectly positively correlated - that is, if ρ_d were equal to +1 - then applying equation (8), we would know that panel income changes in dollars would necessarily be divergent (i.e., $\beta_d > 1$). However, if initial and final incomes are positively correlated but not perfectly so (i.e., $0 < \rho_d < 1$), room is left open for the possibility that a growing economy with rising income inequality might also have convergent dollar changes. Moreover, equation (8) also tells us that the smaller is ρ_d, the more room there is for positive economic growth, rising income inequality, and convergent dollar changes to coexist.

Some analysts may implicitly be supposing that income recipients who are high (low) to begin with will inevitably be high (low) at a later point in time. Whether or not this is the case is an empirical question. The answer should not, however, be assumed.

If during periods of economic decline, the dollar losses of the poor are larger than those of the rich, i.e., if there is divergence in dollars, then the income share of the rich will grow and so will inequality. Hence, in this case it is impossible to have a falling CV together with divergent dollar changes.

What if economic growth is positive and dollar changes are divergent? In that case the dollar gains of the initially poor can be smaller than those of the initially rich, yet the share gains of the anonymous poor can be higher than the share gains of the anonymous rich, leading to a fall in relative inequality.

Our example from the Introduction illustrates this. Namely, for transition [5, 20] → [7, 23], it is the case that $\delta_d = 0.067$, yet the CV falls by 0.067.

In more precise terms, as with any relative inequality index, the coefficient of variation is independent of the measurement scale of income; yet the coefficient of a dollar-change regression is affected by proportional dollar-changes. Even when relative inequality is falling, if positive economic growth is strong enough, it can generate divergence in dollars by proportionally increasing incomes by $1 + g$.

3.2 Inequality Measures and Panel Changes in Shares and Proportions

Next we relate commonly used inequality measures to regressions of share and proportional panel changes. If the only change in incomes between the initial and final periods were a uniform rescaling by a given factor, then the income shares would remain unaltered, and the proportional changes would be constant for all individuals. In this case, our share and proportional change regressions would register coefficients equal to zero (i.e. neutral panel changes). Similarly, all relative inequality measures would remain unaltered. In other words, both relative inequality measures and these regressions have in common an invariance to economy-wide rescalings of income, and because of this natural connection, all the propositions in this section share a common structure.

As we will show, it is always the case that a fall in inequality, as gauged by a particular index, leads to convergence in a specified regression of share or proportional panel changes. Intuitively, this means that for each inequality index we can find a panel regression such that if the income shares of the *anonymous* rich and poor get closer together (i.e. if relative inequality falls), then either the income shares of the *initially* rich and poor also approach one another or the proportional income changes are larger for initially poorer individuals. The counterpart of this result is that if the individuals that started ahead experience larger share or proportional income gains as they go from one period to the next (i.e. if there is divergence), then inequality must rise, as judged by a specific relative inequality index.

A second property present in these propositions is that in order to observe rising relative inequality together with convergent panel changes, these panel changes need to be "large enough". As we will see, what constitutes "large" panel income changes depends on the proposition in question.

We turn now to specific inequality measures, beginning with measures that are related to variance conditions. The reason to begin with this family of inequality indices is that variances can naturally be related to regression coefficients, which of course have variances in their formulas. Furthermore, variance-based inequality measures are widely used in the literature. As previously mentioned, in the macro and labor literatures, it is common to assess changes in relative inequality by focusing on the variance of log-incomes, in spite of its Lorenz-inconsistency. In addition, it is easy to show that the variance of shares is the square of the coefficient of variation (see Lemma S1 in the Supplementary file), which is a Lorenz-consistent inequality measure.

Our first result links the variance of any monotonically increasing function of income in dollars $y = f(d)$ (e.g. logarithms, shares) with the coefficient of a regression of the changes in this generic variable y on its initial level y_0.

Proposition 2 ***Changes in Variance and Convergence for the Class of Monotonic Transformations of Income in Dollars***
Let $f(d)$ be any monotonically increasing function of income in dollars and denote the value of this function by y. Let $V(y)$ denote the variance of y. Finally, let δ_y be defined by the change-on-initial regression

$$\Delta y = \gamma_y + \delta_y y_0 + u_y. \tag{3.2}$$

Then:

i) Falling inequality and convergent panel changes
If the variance of y falls, then the regression of changes in y on initial y is convergent, i.e., $\delta_y < 0$.

ii) Rising inequality and convergent panel changes
If the variance of y rises and if $\Delta V(y) < V(\Delta y)$, then the regression of changes in y on initial y is convergent, i.e., $\delta_y < 0$.

iii) Rising inequality and divergent panel changes
If the variance of y rises and if $\Delta V(y) \geq V(\Delta y)$, then the regression of changes in y on initial y is weakly divergent, i.e., $\delta_y \geq 0$.

iv) Falling inequality and divergent panel changes
It is impossible to simultaneously have falling inequality, i.e., $\Delta V(y) < 0$, and weakly divergent panel changes, i.e., $\delta_y \geq 0$.

The following corollary is the contrapositive of part i) of the above Proposition.[10]

Corollary 1 *If the regression of changes in y on initial y is weakly divergent, i.e., $\delta_y \geq 0$, then the variance of y must rise.*

Proposition 2 and its corollary show that a falling variance of y implies convergent changes of y, or alternatively that weakly divergent panel changes of a monotonically increasing function of dollars $y = f(d)$ implies a rising variance of this function. However, convergence does not imply a falling variance: $\delta_y < 0 \nRightarrow \Delta V(y) < 0$.

These results pertain to any monotonically increasing function of income, as long as we use the same function $y = f(d)$ as dependent and independent variables, i.e. as long as we run share-changes on initial shares, log-dollar changes on initial log-dollars, etc. As previously mentioned, a particular case of this result for the variance of logs and the coefficient in a log-change regression, was derived independently by Furceri (2005) and Wodon and Yitzhaki (2006).

It is instructive to illustrate how this proposition operates in a simple two-person example. Consider in particular an economy in which the anonymous distribution of income in log-dollars changes from (1,3) to (1,5). The underlying panel possibilities are:

$$\begin{aligned} \text{Case I}: &\quad [1,3] \rightarrow [1,5] \\ \text{Case II}: &\quad [1,3] \rightarrow [5,1]. \end{aligned}$$

In Case I, log-dollar changes are divergent, and the variance of log-dollars rises, so we are in part *iii*) of the above proposition. In Case II, however, the increase in the variance of log-dollars $\Delta V(\log d)$ equals 6, while the variance of log-dollar changes, $V(\Delta \log d)$ equals 18. This puts us in part *ii*) of the above proposition, where convergent log-dollar changes co-exist with a rising variance of log-dollars.

[10] Strictly speaking the corollary is the contrapositive of part *i)* of the proposition when the variance falls *or remains constant*, i.e. when $V(y) \leq 0$. Following the arguments in the proof of the proposition it is easy to show that a constant variance of y implies convergence, provided that Δy is not zero for at least one person. However, as we mentioned before, save for Section 3.5, we will exclusively deal with cases when inequality changes from one period to the next. In what follows we will present similar corollaries for each proposition.

The mathematical condition listed in part *ii*) of the proposition specifies in what sense panel income changes need to be "large", so as to have rising inequality together with convergence. More specifically, it establishes that if panel changes are large enough so that their variance is larger than the rise in $V(y)$, then it is possible to reconcile rising inequality (as measured by the variance of y), together with convergent income changes.

Applied to our selected regressions, the results from the above proposition relate convergence coefficients to two commonly used relative inequality measures: the variance of log-dollars and the variance of shares. However, of these, only the variance of shares, is Lorenz-consistent, due to its connection with the coefficient of variation. Thus, Proposition 2 gives us a link between changes in the coefficient of variation and a regression of panel share changes.[11]

Next we turn to a result relating changes in the Gini to share convergence relative to initial ranks r_0.

Proposition 3 ***Changes in the Gini and Convergence of Share-Changes-on-Ranks***

Let G denote the Gini index. Let λ_r be defined by the share-changes-on-rank regression

$$\Delta s = \kappa_r + \lambda_r r_0 + e_r$$

Then:

i) Falling Gini and convergent share-changes-on-ranks

If the Gini falls, then the regression of share-changes on initial ranks is convergent, i.e., $\lambda_r < 0$.

ii) Rising Gini and convergent share-changes-on-ranks

If the Gini rises and if $\Delta G < 2E(s_1 \Delta r)$, then the regression of share-changes on initial ranks is convergent, i.e., $\lambda_r < 0$.

iii) Rising Gini and divergent share-changes-on-ranks

If the Gini rises and if $\Delta G \geq 2E(s_1 \Delta r)$, then the regression of share-changes on initial ranks is weakly divergent, i.e., $\lambda_r \geq 0$.

[11] For completeness, we state that result in the Supplementary file.

iv) Falling Gini and divergent share-changes-on-ranks

It is impossible to simultaneously have a falling Gini, i.e., $\Delta G < 0$, and weakly divergent share-changes-on-ranks, i.e., $\lambda_r \geq 0$.

Corollary 2 *If the regression of share-changes on initial ranks is weakly divergent, i.e., $\lambda_r \geq 0$, then the Gini must weakly rise.*

In the above Proposition, convergence and rising inequality will be observed if the change in the Gini is smaller than a weighted sum of rank changes.[12]

We also can establish a connection between the family of Generalized Entropy indices and share and proportional change regressions. These results are presented next.[13]

Proposition 4 ***Changes in Generalized Entropy Indices with Parameter $\alpha \neq 0$ and Convergence of Share-Changes-on-y***

Let $GE(\alpha \neq 0)$ denote the Generalized Entropy measure with parameter $\alpha \neq 0$, which may take any real value, i.e.

$$GE(\alpha \neq 0) = \begin{cases} (\alpha(\alpha-1)n)^{-1} \sum_{i=1}^{n} [s_{it}^{\alpha} - 1] & \text{if } \alpha \neq 0, 1 \\ n^{-1} \sum_{i=1}^{n} (s_{it} \log s_{it}) & \text{if } \alpha = 1 \end{cases}$$

Let y^e ("e" for "entropy") be a monotonically increasing transformation of income defined by

$$y^e = \begin{cases} \frac{s^{\alpha-1}}{\alpha-1} & \text{if } \alpha \neq 0, 1 \\ \log s & \text{if } \alpha = 1 \end{cases}$$

Let λ_y be defined by the share-change-on-y regression

$$\Delta s = \kappa_y + \lambda_y y_0^e + e_y.$$

Then:

i) Falling $GE(\alpha \neq 0)$ and convergent share-changes-on-y

If the Generalized Entropy index, $GE(\alpha \neq 0)$, falls, then the share-changes-on-y regression is convergent, i.e., $\lambda_y < 0$.

[12] The term $E(s_1 \Delta r)$ in part *ii)* is a weighted sum of rank changes, because it equals $\sum \omega \Delta r$ for weight $\omega = s_1/n$.

[13] It is worth remembering that Theil's First Measure coincides with the Generalized Entropy index when $\alpha = 1$. Thus, the next proposition includes results that apply to Theil's First Measure.

ii) Rising $GE(\alpha \neq 0)$ and convergent share-changes-on-y
If the Generalized Entropy index, $GE(\alpha \neq 0)$, rises, and if

$$\alpha \Delta GE(\alpha \neq 0) < E(s_1 \Delta y^e),$$

then the share-changes-on-y regression is convergent, i.e., $\lambda_y < 0$.

iii) Rising $GE(\alpha \neq 0)$ and divergent share-changes-on-y
If the Generalized Entropy index, $GE(\alpha \neq 0)$, rises, and if

$$\alpha \Delta GE(\alpha \neq 0) \geq E(s_1 \Delta y^e),$$

then the share-changes-on-y regression is weakly divergent, i.e., $\lambda_y \geq 0$.

iv) Falling $GE(\alpha \neq 0)$ and divergent share-changes-on-y
It is impossible to simultaneously have a falling Generalized Entropy, i.e., $\Delta GE(\alpha \neq 0) < 0$, and weakly divergent share-changes-on-y, i.e., $\lambda_y \geq 0$.

Corollary 3 *If the share-changes-on-y regression is weakly divergent, i.e., $\lambda_y \geq 0$, then the Generalized Entropy index $GE(\alpha \neq 0)$ must weakly rise.*

In the above Proposition, we can reconcile a rising GE index with share convergence as long as panel share changes are "large" in the sense that $\alpha \Delta GE(\alpha \neq 0) < E(s_1 \Delta y^e)$. Namely, as long as the rise in the inequality index is less than a weighted sum of the panel changes in our transformed income variable y^e. Next, we present the corresponding result for the case of Generalized Entropy index with parameter $\alpha = 0$.

Proposition 5 ***Changes in the Generalized Entropy Index with parameter $\alpha = 0$ and Convergence of Exact Proportional Changes***
Let $GE(0)$ denote the Generalized Entropy measure with parameter $\alpha = 0$, i.e.,

$$GE(0) = n^{-1} \sum_{i=1}^{n} \log(1/s_{it})$$

Let θ be defined by the exact proportional changes regression

$$(d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}. \tag{5}$$

Then:

i) Falling $GE(0)$ and convergent exact proportional changes

If the Generalized Entropy index with parameter $\alpha = 0$, $GE(0)$, falls, then the exact proportional changes regression is convergent, i.e., $\theta < 0$.

ii) Rising $GE(0)$ and convergent exact proportional changes

If

$$E(\Delta \log s) < 0 < E\left(\frac{s_1 - s_0}{s_0}\right),$$

then the Generalized Entropy index with parameter $\alpha = 0$, $GE(0)$, rises, and the exact proportional changes regression is convergent, i.e., $\theta < 0$.

iii) Rising $GE(0)$ and divergent exact proportional changes

If both $E(\Delta \log s)$ and $E\left(\frac{s_1 - s_0}{s_0}\right)$ are negative, then the Generalized Entropy index with parameter $\alpha = 0$, $GE(0)$, rises, and the exact proportional changes regression is divergent, i.e., $\theta > 0$.

iv) Falling $GE(0)$ and divergent exact proportional changes

It is impossible to simultaneously have a falling Generalized Entropy index with parameter $\alpha = 0$, i.e., $\Delta GE(0) < 0$, and a weakly divergent exact proportional changes regression, i.e., $\theta \geq 0$.

Corollary 4 *If the exact proportional changes regression is weakly divergent, i.e., $\theta \geq 0$, then the Generalized Entropy index with parameter $\alpha = 0$, $GE(0)$, must weakly rise.*

In part *ii)* of the above Proposition, the condition expressing that panel income changes need to be large enough is that the average exact proportional share changes $E[(s_1 - s_0)/s_0]$ have the opposite sign to the average change in log shares, $E(\Delta \log s)$. Changes in logs are an approximation to exact proportional changes, provided that the latter are small. Hence, when these two entities have opposite signs, it follows that share changes in the panel are large.

There is a monotonic relationship between the Generalized Entropy family and the Atkinson family of inequality measures. Thus, conditions for the Aktinson index follow directly from the preceding results for the GE family. These results are included in the Supplementary file.

To conclude this section on inequality indices, note that the impossibilities in the above Propositions occur only when we pair up a given inequality index with the appropriate panel income change regression. Absent the appropriate pairing, falling inequality and divergent panel income changes can both arise.

We now turn to results linking our income change regressions to changes in inequality under Lorenz dominance.

3.3 Lorenz Dominance and Panel Income Changes

In spite of the wide use of the indices analyzed in the previous section, the Lorenz Dominance criterion remains the most widely accepted way of judging whether relative inequality has risen or fallen. The reason for this is that whenever this criterion provides an ordering of the inequalities of two distributions, all Lorenz-consistent indices agree with that ordering. In other words, when for two income distributions A and B the Lorenz criterion deems $A \succ B$, it will be the case that all Lorenz-consistent inequality indices deem distribution B to be more unequal.

It turns out that we can find a set of useful results linking Lorenz Dominance to our previous regression methods. We present those results next. It then follows that all these results also apply to the family of Lorenz-consistent inequality indices whenever the Lorenz curves of distributions A and B do not cross.

3.3.1 Lorenz Dominance and Share Changes

In this section we derive a connection between the Lorenz Dominance criterion

$$\begin{aligned} & s_{1c} + s_{2c} + \ldots + s_{jc} \geq s_{10} + s_{20} + \ldots + s_{j0} \quad \text{for j = 1, 2, \ldots, n - 1 and} \\ & s_{1c} + s_{2c} + \ldots + s_{jc} > s_{10} + s_{20} + \ldots + s_{j0} \quad \text{for some j < n} \end{aligned} \tag{2}$$

and a share-change regression

$$\Delta s = \gamma_s + \delta_s s_0 + u_s. \tag{9}$$

Equations (2) and (9) both involve initial and final income-shares. However, the final period shares appear sorted differently in the two expressions. More specifically, in condition (2), final shares s_c are sorted in ascending order of *final* shares, while in equation (9) final shares s_1 preserve the order of *initial* shares.

It is easy to show that the sign of the coefficient δ_s in regression (9) is determined by the sign of the covariance

$$cov(\Delta s, s_0) = \frac{\sum_i (s_{i1} - s_{i0}) s_{i0}}{n},$$

since average share changes are zero by construction.

Using vector s_c as defined in (1), we can decompose this covariance as

$$cov(\Delta s, s_0) = \frac{\sum_i [(s_{i1} - s_{ic}) + (s_{ic} - s_{i0})] s_{i0}}{n}.$$

That is, whether share changes are convergent or divergent is determined by the sum of two terms, a structural mobility term and an exchange mobility term:

$$\begin{aligned} SM &= \frac{\sum_i (s_{ic} - s_{i0}) s_{i0}}{n} \\ XM &= \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n}. \end{aligned} \tag{10}$$

SM captures the component of the covariance associated with changes in the shape of the income distribution for anonymous people, and XM is the component of the covariance associated with positional change, under a fixed marginal distribution.[14]

We can derive the following two key Lemmas for these terms.

Lemma 1 *Let SM be given by equation (10), then:*

i) A Lorenz-improvement, $LC_1 \succ LC_0$*, implies* $SM < 0$.

ii) A Lorenz-worsening, $LC_1 \prec LC_0$*, implies* $SM > 0$.

In other words, in cases of Lorenz-dominance, the sign of SM fully reflects whether there has been a fall or a rise in inequality judged by the Lorenz-criterion.

As previously mentioned, when looking at income changes, we care not only about how the anonymous distribution of income evolves, but also about who moved to a different position across periods. This is reflected by the transition from s_c to s_1. In this transition, share changes will be convergent, since in the reranking of individuals there will always be a positive transfer of income shares from a relatively richer individual to a poorer one. This is expressed in the following Lemma.

[14]This is so because if positions were to remain unchanged, i.e. $s_c = s_1$, the entire share change would be due to a change in the shape of the distribution, $s_c - s_0$.

Lemma 2

$$XM = \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n} \leq 0.$$

With these two results we can proceed to analyze the connection between share mobility and changes in inequality as measured by Lorenz comparisons.

Proposition 6 ***Lorenz Dominance and Convergence in Shares***
Let δ_s be defined by the share-change regression

$$\Delta s = \gamma_s + \delta_s s_0 + u_s. \tag{9}$$

Then:

i) Falling inequality and convergent share changes
If there is a Lorenz-improvement, $LC_1 \succ LC_0$, then the regression of share changes on initial shares is convergent, i.e., $\delta_s < 0$.

ii) Rising inequality and convergent share changes
If there is a Lorenz-worsening, $LC_1 \prec LC_0$ and if $|XM| > SM$, then the regression of share changes on initial shares is convergent, i.e., $\delta_s < 0$.

iii) Rising inequality and divergent share changes
If there is a Lorenz-worsening, $LC_1 \prec LC_0$ and if $|XM| \leq SM$, then the regression of share changes on initial shares is weakly divergent, i.e., $\delta_s \geq 0$.

iv) Falling inequality and divergent share changes
It is impossible to simultaneously have a Lorenz-improvement, $LC_1 \succ LC_0$, and weakly divergent share changes, i.e., $\delta_s \geq 0$.

Corollary 5 *If the regression of share changes on initial shares is weakly divergent, i.e., $\delta_s \geq 0$, then either a weak Lorenz-worsening has taken place, $LC_1 \preceq LC_0$, or the Lorenz curves of incomes in periods 0 and 1 cross.*

The intuition (and proof) behind this proposition is related to a well-known result in the inequality literature stating that an equalization in the Lorenz sense can be achieved by a series of income transfers from richer to poorer individuals that keep unaltered the individual ranks between the initial and the final periods (see for instance Fields and Fei, 1978).

These progressive transfers generate by construction convergent share changes in the transition from s_0 to s_c (Lemma 1). However, when going from s_0 to s_1, we also need to consider the transition from s_c to s_1. In this last step the shape of the income distribution remains unchanged and pairs of individuals swap incomes and therefore positions. As we saw in Lemma 2, this positional rearrangement leads to convergent share changes always.

Hence, in the case of a Lorenz-improvement, both XM and SM go in the same direction, and share changes are convergent. However, in the case of a Lorenz-worsening, the two components will move in opposite directions, and depending on which force is dominant there will be convergence or divergence in shares as measured by δ_s in equation (9).

In contrast, if all individuals keep their same rank in the initial and final distributions (i.e. if there is zero positional mobility), vector s_c will equal the final share vector s_1, and the sign of δ_s is determined exclusively by SM. Given Lemma 1 and the connection between SM and δ_s, in the absence of positional changes, we have that a Lorenz-worsening leads to divergent share changes.

In other words, as long as we restrict ourselves to the case of no positional mobility and no crossings of Lorenz curves, share mobility and changes in inequality fully align, in the sense that rising inequality as gauged by Lorenz-worsening only occurs with divergent share-changes and falling inequality as gauged by Lorenz-improvement only occurs with convergent share-changes. If individuals swap positions from one period to the next, the direction of the inequality change and divergence/convergence need not align one-to-one.

As happened with the propositions in section 3.2, when panel income changes are large (and in the right direction), there can be convergence together with rising inequality. In this case, the condition expressing "large" panel changes is that the exchange-mobility component $|XM|$ is larger than the structural-mobility one SM. This is Part *ii)* of Proposition 6.

Finally, Corollary 5 expresses the idea that when share changes are divergent, the income shares of the initially rich grow relative to others' shares (irrespective of whether there is positional change or not). This should lead to disequalization. Hence, the only possible way to register a fall in inequality in this instance is for Lorenz curves to cross.[15]

[15] As is well known, when Lorenz curves cross, a Lorenz-consistent measure can always be found showing rising inequality and another Lorenz-consistent measure can be found showing falling inequality.

3.3.2 Lorenz Dominance and Proportional Income Changes

We next explore the relationship between proportional changes in income and Lorenz-improvement/worsening.

Log-Income Approximation

The most common way to measure proportional convergence is by approximating proportional changes by differences in log-income and estimating a double-log regression

$$\Delta \log d = \gamma_{\log} + \delta_{\log} \log d_0 + u_{\log} \tag{11}$$

or its equivalent final-on-initial form $\log d_1 = \alpha_{\log} + \beta_{\log} \log d_0 + u_{\log}$. As we now show, these types of regressions have misleading properties.

Consider the following example:

$$[1, 1, 1, 1, 1, 1, 1, 1, 6, 9] \rightarrow [1, 1, 1, 1, 1, 1, 1, 1, 7, 8].$$

The richest person has transferred \$1 to the next richest person, which is a clear Lorenz-improvement. Inequality therefore falls by the Lorenz criterion and accordingly for any Lorenz-consistent inequality measure. Moreover, a rank-preserving transfer in dollars from the richest person to anyone lower down in the income distribution should be deemed convergent, as it brings convergence in dollars (in this case $\delta_d = -0.04$). However, if in this example, we regress the change in log-dollars on initial log-dollars, we obtain $\delta_{\log} = 0.00045$, and hence find divergence in log-dollars. Thus, in this example, a Lorenz-improvement has taken place and yet the regression of log-income changes on initial log-income registers divergence.[16]

The previous example illustrates a more general point: that log-incomes can be divergent if a progressive transfer occurs sufficiently high-up in the income distribution.

More precisely, we can show the following result for a single rank-preserving transfer that is sufficiently small:

Proposition 7 ***Lorenz Dominance and Log-income Panel Changes under a Single Rank-Preserving Transfer Sufficiently High Up in***

[16] In addition, the variance of log-incomes increases, which it must by Corollary 1 applied to logarithms.

the Income Distribution

Let δ_{log} be defined by the regression

$$\Delta \log d = \gamma_{log} + \delta_{log} \log d_0 + u_{log}. \quad (11)$$

Furthermore, let gm_0 denote the geometric mean of income in period 0, and note that $\exp(1) \approx 2.718$ *(Euler's number). Consider two individuals i and j such that $d_{i0} > d_{j0} > gm_0 * \exp(1)$. Let $h > 0$ be a small rank-preserving transfer between i and j. Then:*

a) If such a transfer h is equalizing, it produces a Lorenz-improvement $LC_1 \succ LC_0$ and a divergent regression coefficient, i.e. $\delta_{log} > 0$.

b) If such a transfer h is disequalizing, it produces a Lorenz-worsening $LC_1 \prec LC_0$ and a convergent regression coefficient, i.e. $\delta_{log} < 0$.

Proposition 7 suggests why it would be easy to misinterpret a log-change regression like (11). The log-change regression can indicate divergence as we define it, even when the income changes lead to a Lorenz-improvement. Rank-preserving equalizations which occur sufficiently high-up in the income distribution can lead to divergence in log-dollars. This is an unappealing property of the log-income change approximation to exact proportional change.[17]

As mentioned before, it is well known in the literature that the variance of log-incomes is not Lorenz-consistent. In fact Cowell (2011) shows that under a transfer similar to the one in Proposition 7, the variance of logarithms will move in the opposite direction to the Lorenz curve, and Foster and Ok (1999) have shown that the Lorenz-inconsistency of this variance can occur under even more general circumstances. Proposition 7 adds then further reasons to be cautious when using log regressions and variance of logs.

Exact Proportional Changes

As previously mentioned, one alternative to the log-income changes regression (11) is to regress the exact proportional change in incomes on initial

[17] Furthermore, as shown in Table S3, we can find all possible combinations of Lorenz-worsening/improvement with convergent/divergent log-income changes. In particular, contrary to the share-change case, we can find examples that make compatible falling inequality as gauged by a Lorenz-improvement and divergent log-income changes.

income as in equation (5). In this case, we can establish results and conditions linking Lorenz-improvements/worsenings with convergent/divergent exact proportional changes. In order to do this it is useful to define terms for *proportional* structural mobility (PSM) and *proportional* exchange mobility (PXM):

$$PSM = \frac{1}{n}\sum_i \frac{s_{ic} - s_{i0}}{s_{i0}}$$
$$PXM = \frac{1}{n}\sum_i \frac{s_{i1} - s_{ic}}{s_{i0}}. \tag{12}$$

Similar to the analysis of share changes, PSM is a term capturing the average proportional share changes due to changes in the shape of the income distribution if positions remain unchanged. In turn, PXM reflects proportional share changes associated with positional rearrangements, under a fixed marginal distribution. We can establish the following lemmas for these two terms, which mirror Lemmas 1 and 2.

Lemma 3 *Let PSM be given by equation (12). Then:*

i) A Lorenz-improvement, $LC_1 \succ LC_0$, implies $PSM > 0$.

ii) A Lorenz-worsening, $LC_1 \prec LC_0$, implies $PSM < 0$.

Lemma 4

$$PXM = \frac{1}{n}\sum_i \frac{s_{i1} - s_{ic}}{s_{i0}} \geq 0.$$

With these lemmas established, we can show the following results linking inequality changes and exact proportional changes.

Proposition 8 ***Lorenz Dominance and Convergence in Exact Proportional Changes***

Let θ be defined by the exact proportional change regression

$$\text{pch d} \equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}. \tag{5}$$

Then:

i) Falling inequality and convergent exact proportional changes

If there is a Lorenz-improvement, $LC_1 \succ LC_0$, then the exact proportional change regression is convergent, i.e., $\theta < 0$.

ii) Rising inequality and convergent exact proportional changes

If there is a Lorenz-worsening, $LC_1 \prec LC_0$, and if $PXM > |PSM|$, then the exact proportional change regression is convergent, i.e., $\theta < 0$.

iii) Rising inequality and divergent exact proportional changes

If there is a Lorenz-worsening, $LC_1 \prec LC_0$, and if $PXM \leq |PSM|$, then the exact proportional change regression is weakly divergent, i.e., $\theta \geq 0$.

iv) Falling inequality and divergent exact proportional changes

It is impossible to simultaneously have a Lorenz-improvement, $LC_1 \succ LC_0$, and weakly divergent exact proportional changes, i.e., $\theta \geq 0$.

Corollary 6 *If the exact proportional change regression is weakly divergent, i.e., $\theta \geq 0$, then either a weak Lorenz-worsening has taken place, $LC_1 \preceq LC_0$, or the Lorenz curves of incomes in periods 0 and 1 cross.*

The intuition of this Proposition is similar to the one in Section 3.2: if income changes are large enough, and in a suitable pattern, we can have positional changes, rising inequality, and convergent proportional changes all taking place at the same time. Also a comparison of Propositions 6 and 8 shows that share-change and exact proportional change regressions have a common structure with respect to Lorenz-dominance.

3.3.3 Lorenz Dominance and Changes in Dollars

While the previous subsections established a clear connection between change in inequality as gauged by the Lorenz criterion and share and proportional panel changes, some evaluators may be interested specifically in panel income changes in dollars. To close this section we establish a condition relating changes in inequality under Lorenz-dominance and a dollar-change regression.[18]

Proposition 9 ***Lorenz Dominance and Convergence in Dollars***
Let β_d and β_s be defined by the regressions

$$d_1 = \alpha_d + \beta_d d_0 + u_d \tag{6}$$

[18] In separate work we establish a connection between panel changes in dollars and the so-called Absolute Lorenz Curves, as defined by Moyes (1999).

and

$$s_1 = \alpha_s + \beta_s s_0 + u_s, \tag{13}$$

respectively. Let g denote the economy-wide growth rate in incomes between year 0 and year 1. Then:

i) Negative growth, falling inequality and convergent dollar changes

If $g \leq 0$, and if there is a Lorenz-improvement, $LC_1 \succ LC_0$, then the final-on-initial dollar regression is convergent, i.e., $\beta_d < 1$.

ii) Negative growth, rising inequality and divergent dollar changes

If $g \leq 0$, and if the final-on-initial dollar regression is weakly divergent, i.e., $\beta_d \geq 1$, then either a weak Lorenz-worsening has taken place, $LC_1 \preceq LC_0$, or the Lorenz curves of incomes in periods 0 and 1 cross.

iii) Negative growth, falling inequality and divergent dollar changes

If $g \leq 0$, it is impossible to simultaneously have a Lorenz-improvement, $LC_1 \succ LC_0$, and weakly divergent dollar changes, i.e., $\beta_d \geq 1$.

iv) For all other combinations of Lorenz-improvement, $LC_1 \succ LC_0$, or worsening, $LC_1 \preceq LC_0$, and for positive or negative growth: the dollar changes are convergent (divergent) as $\beta_s(1+g)$ is smaller (greater) than one.

Similar to what we found in Section 3.1, this proposition indicates that it is impossible to reconcile a Lorenz-improvement and divergent dollar changes under a scenario with negative income growth. However, it is possible to simultaneously have a Lorenz-improvement and divergence in dollars provided that the average income growth in the economy is positive and strong enough. For an illustration remember our example from the introduction of the paper, namely the transition $[5, 20] \rightarrow [7, 23]$.

It is easy to establish that the regression coefficient β_d equals $\beta_s(1+g)$ (see Lemma S3 in the Supplementary file). Part *iv)* of Proposition 9 implies that in order to have a Lorenz-worsening and convergent dollar changes, under a scenario of positive income growth, $g > 0$, the condition $\beta_s(1 + g) < 1$ must hold – that is, we need sufficiently strong convergence in shares. However, in Proposition 6.*ii)* we saw that, for XM and SM defined in equation (10), share convergence and a Lorenz-worsening can only occur when $|XM| > SM$.

This in turn implies that income changes need to be large enough so that some individuals change positions in a widening distribution.

In contrast, in periods of economic decline, we can have convergent dollar changes together with a Lorenz-worsening, even when all individuals keep their initial position.[19]

3.4 Extensions to Cases Involving Single Lorenz Crossings

As previously noted, all the results in section 3.3 were derived by analyzing rising or falling inequality as judged by Lorenz-worsenings or improvements. Of course, it is possible for the Lorenz curves of two distributions to cross, which often happens in practice.[20] How far can we go when Lorenz curves cross? In addition to our results for specific inequality indices, which still apply, it is also possible to establish more general results for the class of "Transfer-Sensitive inequality indices" (Shorrocks and Foster, 1987). These indices allow certain pairs of distributions to be ranked in the presence of Lorenz-crossings by giving greater weight to transfers that occur in the lower part of the income distribution. More specifically this family of indices can rank two distributions when one distribution third-order stochastically dominates the other one.[21]

In section S.4 in the Supplement file we present a result linking the class of Transfer Sensitive indices with the share change regression (9).

3.5 Special Cases

To conclude this section we briefly discuss three special types of changes in the income distribution vector to gain additional understanding of how different patterns of Lorenz curve changes and divergent/convergent panel income changes can arise.

[19] In this case, the relative income shares of the initially rich increase, in spite of the fact that their dollar losses are larger than those of the rest of the population. For an example, consider the opposite transition $[7, 23] \rightarrow [5, 20]$.

[20] See Atkinson (1973, 2008) for a classic discussion of the available evidence on Lorenz-crossings using real data in a cross-country setting.

[21] Shorrocks and Foster (1987) show that the Atkinson family and the Generalized Entropy class with $\alpha < 2$ satisfy the transfer-sensitive property, but the Gini coefficient does not.

First, we consider the case where the anonymous income distribution vector does not change, but where individuals swap positions. Since there is no change in the anonymous distribution, the Lorenz curves and all inequality indices will remain unchanged. However, the positional swaps that occur will lead to convergence in our panel regressions no matter how we measure it.

The second special case we consider is one in which all incomes change proportionally, i.e. all incomes are scaled-up or down by a constant multiplicative factor κ. In this case, all relative inequality measures will remain the same, also the panel income changes are recorded as neutral, i.e. neither convergent nor divergent, by three of our panel regressions. The one exception arises in the case of the dollar change regression, because a uniform proportional increase(decrease) in dollars makes the dollar gains(losses) of the initially rich larger than those of the initially poor.

Finally, we consider the case when all individuals keep their same positions and yet there is a Lorenz-worsening. This case is of interest because, as we saw in the previous sections, it is not possible to rule out the existence of convergent panel income changes when inequality rises, due to the fact that there might be crossings among panel people as we go from one period to the other. It is then interesting to see whether in the absence of positional changes, rising relative inequality is or is not a sufficient condition for divergent panel changes. Furthermore, this scenario of a Lorenz-worsening with no positional change, seems to be what many people have in mind when they think of increases in inequality.

In this third special case, most of our regressions record divergent panel changes. However, to get divergence in dollar changes, we need to assume the additional condition that economic growth has been non-negative.[22] Also, in the case of the log-change regression, we have an ambiguous result depending on where in the distribution the disequalizing income changes are taking place (see Proposition 7).[23]

The results for the three special cases are summarized and established in section S.5 of the Supplementary file.

This concludes our derivation of results. We turn now to a summary of

[22]Under negative growth we could find convergent, neutral or divergent dollar changes depending on the magnitude of the growth rate of income, g.

[23]This conclusion does not extend to cases of rising relative inequality, without Lorenz-worsening. For instance, in the case of the transition $[1, 2.99, 3] \rightarrow [2, 2.001, 5]$, there is a rise in inequality (the CV goes from 0.40 to 0.47), the Lorenz curves cross, there is no positional change, yet all of our regressions register convergence, rather than divergence.

the results and a concluding discussion.

4 Summary of Results and Concluding Observations

This paper has explored mathematically the relationship between changing relative income inequality in the cross section and panel income changes. In spite of all four combinations - rising inequality and convergent panel income changes, rising inequality and divergent panel income changes, falling inequality and convergent panel income changes, and falling inequality and divergent panel income changes - being possible under scenarios of positive and negative aggregate income growth, we find that for each way of assessing change in inequality, there is a corresponding panel income change regression, such that two intuitive properties hold: i) if the income distribution becomes less unequal relatively then there is convergence in panel incomes, and ii) if panel income changes are divergent, then relative inequality rises. Intuitively, the first property says that if the anonymous rich and the anonymous poor get closer together relatively when going from period 0 to period 1, then the identified rich and poor cannot be farther apart in relative terms. The second property means that if the identified initially rich and poor move relatively farther away from one another, then the anonymous income distribution must become relatively more unequal. In our paper, we show that these intuitive properties hold for arbitrary n-person economies, for a wide variety of ways of measuring inequality, provided that we pair each relative inequality measure with the appropriate relative panel income-change regression.

Dollar-change regressions are interesting, because they indicate how much additional purchasing power each recipient gains in the course of economic growth. During periods of strong positive economic growth, there can be falling relative inequality, convergent panel income changes in relative terms, and yet divergent dollar changes. The two preceding intuitive properties are not violated by dollar change regressions precisely because dollar change regressions are not relative.

What may be problematical is the regression of log-dollar changes on initial log-dollars when there is Lorenz-dominance. In particular, we have shown that log-dollar change regressions, like the ones commonly used in the macro and labor literatures, have the unappealing property of deeming panel income changes divergent when rank-preserving equalizing transfers occur sufficiently high-up in the income distribution.

To conclude let us return to where we started; namely with the reconciliation between i) convergent panel income changes and rising inequality, and between ii) divergent panel income changes and falling inequality.

Convergence can occur in spite of rising inequality if panel income changes are large enough such that some initially low-earners become high earners in a widening distribution. In fact, it is precisely because panel studies abandon the property of single-period anonymity and replace it by two-period anonymity that such a pattern can be identified. In particular, our paper shows how it is possible to have convergent dollar changes, together with rising relative inequality measures, in times of economic growth - a combination that is often observed in empirical data. In addition, the coexistence of divergent panel income changes and falling inequality depends crucially on the way inequality and divergence are measured.

The results derived in this paper open up additional questions as to the empirical nature of individual income changes. For instance, when rising inequality is observed together with convergent panel income changes in empirical work, is this finding driven by a few individuals experiencing large changes, by many individuals experiencing moderate changes, or are both important? Exploring the precise way in which these individual income changes occur is an important question for future research.

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Supplement to "Panel Income Changes and Changing Relative Income Inequality"

Robert Duval Hernández[1], Gary S. Fields[*2] and George H. Jakubson[3]

[1]Open University of Cyprus, UCY, CIDE, and IZA
[2]Cornell University, IZA, and WIDER
[3]Cornell University

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[*]**Corresponding Author:** gsf2@cornell.edu

S.1 Some useful Lemmas

Before presenting the proofs of the results in the paper we include a set of Lemmas that will be used in establishing the subsequent results.

Lemma S1. ***Variance of Shares and Coefficient of Variation***
Let $CV(d)$ denote the coefficient of variation of income, then

$$CV^2(d) = V(s).$$

Proof of Lemma S1.

$$V(s) = V\left(\frac{d}{\mu}\right) = \frac{1}{\mu^2}V(d) = CV^2(d).$$

□

Lemma S2. ***Share Changes and Exact Proportional Changes***
Let θ be defined by the exact proportional change regression

$$\text{pch d} \equiv (d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}. \tag{5}$$

Then:

$$sgn(\theta) = -sgn\left(E\left[\frac{s_1 - s_0}{s_0}\right]\right).$$

Proof of Lemma S2. First, rewrite the proportional change regression (5) as

$$\frac{d_1}{d_0} = (\phi + 1) + \theta d_0 + u_{pch}.$$

Then the sign of θ will depend on the sign of the covariance

$$\begin{aligned} cov\left(\frac{d_1}{d_0}, d_0\right) &= E\left(\frac{d_1}{d_0}d_0\right) - E\left(\frac{d_1}{d_0}\right)\mu_0 \\ &= \mu_1 - E\left(\frac{d_1}{d_0}\right)\mu_0. \end{aligned}$$

Hence, there will be divergence (i.e. $\theta > 0$) whenever $\mu_1 > E(\frac{d_1}{d_0})\mu_0$, convergence (i.e. $\theta < 0$) whenever $\mu_1 < E(\frac{d_1}{d_0})\mu_0$, otherwise the profiles will be parallel.

This condition for convergence can be re-expressed as

$$
\begin{aligned}
E\left(\frac{d_1}{d_0}\right)\mu_0 - \mu_1 > 0 \iff \\
E\left(\frac{d_1}{d_0}\right)\frac{\mu_0}{\mu_1} - 1 > 0 \iff \\
E\left(\frac{s_1}{s_0}\right) - 1 > 0.
\end{aligned}
$$

So we can express these conditions as:
Convergence ($\theta < 0$) $\iff$ $0 < E[\frac{s_1 - s_0}{s_0}]$
Divergence ($\theta > 0$) $\iff$ $0 > E[\frac{s_1 - s_0}{s_0}]$
Parallel Profiles ($\theta = 0$) $\iff$ $0 = E[\frac{s_1 - s_0}{s_0}]$ □

Lemma S3. *Let μ_t denote the mean income in period t, β_d and β_s denote the convergence coefficients of regressions (6) and (13) in dollars and in shares, respectively, and g denote the economy-wide growth rate in incomes between year 0 and year 1. Then*

$$\beta_d = \beta_s \frac{\mu_1}{\mu_0} = \beta_s(1+g).$$

Proof of Lemma S3. The final-on-initial regression in dollars (6) is

$$d_1 = \alpha_d + \beta_d d_0 + u_d.$$

Dividing this equation by μ_1 we obtain

$$
\begin{aligned}
s_1 &= \frac{\alpha_d}{\mu_1} + \beta_d \frac{d_0}{\mu_1} + \frac{u_d}{\mu_1} \\
&= \frac{\alpha_d}{\mu_1} + \beta_d \frac{\mu_0}{\mu_1}\frac{d_0}{\mu_0} + \frac{u_d}{\mu_1} \\
&= \frac{\alpha_d}{\mu_1} + \beta_d \frac{\mu_0}{\mu_1} s_0 + u_s.
\end{aligned}
$$

Hence, for the final-on-initial regression in shares (13)

$$s_1 = \alpha_s + \beta_s s_0 + u_s$$

we have that,

$$\alpha_s = \frac{\alpha_d}{\mu_1}; \beta_s = \beta_d \frac{\mu_0}{\mu_1} = \beta_d \frac{1}{1+g}.$$

The Lemma follows from this last equation.

□

S.2 Proofs of Results in the Main Text

Proof of Proposition 1. By definition

$$\rho_d = \frac{cov(d_1, d_0)}{\sqrt{V(d_1)}\sqrt{V(d_0)}}$$

and

$$\beta_d = \rho_d \frac{\sqrt{V(d_1)}}{\sqrt{V(d_0)}}.$$

However,

$$\frac{\sqrt{V(d_1)}}{\sqrt{V(d_0)}} = \frac{\sqrt{V(d_1)}/\mu_1}{\sqrt{V(d_0)}/\mu_0} \frac{\mu_1}{\mu_0} = \frac{CV(d_1)}{CV(d_0)} \frac{\mu_1}{\mu_0}.$$

Moreover,

$$\mu_1 = (1+g)\mu_0$$

where g is the economy-wide income growth rate. Combining these equations together we obtain equation (8). □

Proof of Proposition 2. Start from the identity

$$y_1 = y_0 + \Delta y,$$

take variances on both sides and rearrange to arrive at

$$\Delta V(y) = V(\Delta y) + 2cov(\Delta y, y_0). \qquad \text{(S.1)}$$

Also recall that whenever $cov(\Delta y, y_0) < 0$, there is convergence in (generic) panel income changes, i.e. $\delta_y < 0$.

Since $V(\Delta y) > 0$, it follows immediately from equation (S.1) that whenever $\Delta V(y) \leq 0$, it must be that $cov(\Delta y, y_0) < 0$, and thus δ_y is negative. This establishes part i).

To establish part ii) notice again from (S.1) that changes in y are convergent (i.e. $cov(\Delta y, y_0) < 0$), as long as $\Delta V(y) < V(\Delta y)$, even if the variance of y is rising.

Part iii) follows from this same equation (S.1), since if $\Delta V(y) \geq V(\Delta y)$ then it must be that $cov(\Delta y, y_0) \geq 0$, and thus $\delta_y \geq 0$.

Finally, the impossibility result from part iv) follows immediately from part i). □

Proof of Proposition 3. We can express the Gini index at time t as

$$G_t = -\frac{n+1}{n} + \frac{2}{n}\sum_{i=1}^{n} r_{it}s_{it}.$$

Hence, the change in Ginis can be expressed as

$$\begin{aligned} G_1 - G_0 &= \frac{2}{n}\sum_{i=1}^{n}(r_{i1}s_{i1} - r_{i0}s_{i0}) \\ &= \frac{2}{n}\sum_{i=1}^{n}(r_{i1}s_{i1} - r_{i0}s_{i1} + r_{i0}s_{i1} - r_{i0}s_{i0}) \\ &= \frac{2}{n}\sum_{i=1}^{n}[(r_{i1} - r_{i0})s_{i1} + r_{i0}(s_{i1} - s_{i0})]. \end{aligned}$$

In other words, we arrive at the following decomposition

$$\begin{aligned} \Delta G &= \frac{2}{n}\sum_{i=1}^{n}(s_{i1}\Delta r_i + r_{i0}\Delta s_i) \\ &= 2[E(s_1\Delta r) + E(r_0\Delta s)] \end{aligned} \tag{S.2}$$

To establish the results in this proposition we will first show that $E(s_1\Delta r)$ is always non-negative. Then we will show that that the sign of λ_r is given by the sign of the second term in (S.2), i.e., $E(r_0\Delta s)$.

The following Lemma establishes the first result.

Lemma.

$$\sum_{i=1}^{n} s_{i1}\Delta r_i \geq 0.$$

$\triangleleft$*Proof of Lemma*

Order the individuals in ascending order of initial income shares, and create an $n \times n$ matrix A, whose rows and columns identify the individuals in that order- same order for rows and columns.

Let the entries of the A matrix be filled as follows: if individual p overtakes individual q (i.e., p's rank is less than q's initially, but greater in final shares), then $A(p,q) = 1$ and $A(q,p) = -1$. That is, $A(p,q) = 1$ if p's share of final income exceeds that of q when p's share of initial income was less than q's. If two individuals do not overtake one another then $A(q,p) = 0$. By constructing the matrix A in this manner we ensure that for all $i, j \in \{1, \dots, n\}$ it is the case that $A(i,j) = -A(j,i)$.

From this construction, it follows that we can express the change in ranks for any given individual as the column sum for a given row of matrix A, i.e.,

$$\Delta r_i = \sum_{q=1}^{n} A(i,q).$$

Hence we can write,

$$\sum_{i=1}^{n} s_{i1}\Delta r_i = \sum_{i=1}^{n} s_{i1}\left(\sum_{q=1}^{n} A(i,q)\right).$$

This sum aggregates terms of the form $s_{i1}A(i,j)$ for three types of pairs (i,j):

a) $Z = \{i, j \in (1, \dots, n) | A(i,j) = 0\}$,

b) $Pos = \{i, j \in (1, \dots, n) | A(i,j) = 1\}$,

c) $Neg = \{i, j \in (1, \dots, n) | A(i,j) = -1\}$.

For each element in the set Pos there is a corresponding element in the set Neg (since $A(i,j) = -A(j,i)$), so we can now sum over the pairs

$$\begin{aligned}\sum_{i=1}^{n} s_{i1}\Delta r_i &= \sum_{(p,q)\notin Z} [s_{p1}A(p,q) + s_{q1}A(q,p)] \\ &= \sum_{(p,q)\notin Z} (s_{p1} - s_{q1})\end{aligned}$$

yet we know that $s_{p1} > s_{q1}$ since p overtook q going up, hence we have a sum of positives (or zeroes if there was no positional change).

This completes the proof of the Lemma.▷

The intuition behind this result is that for any upward rank change there will be one or more downward rank changes such that the overall sum of the upward and downward rank changes is zero. The upward rank change is multiplied by a larger *final* income share than are the downward rank changes. This is true for all upward rank changes, individually and together.

As a consequence of this last Lemma and equation (S.2) we can establish that if $\Delta G < 0$, it must be that the second term

$$\frac{2}{n}\sum_{i=1}^{n} r_{i0}\Delta s_i = 2E(r_0\Delta s)$$

is negative.

This term however, is a (rescaled) covariance between share changes and initial ranks. In particular,

$$\begin{aligned} cov(\Delta s, r_0) &= E(r_0\Delta s) - E(r_0)E(\Delta s) \\ &= E(r_0\Delta s), \end{aligned}$$

as $E(\Delta s) = 0$, by construction. If this term is negative (as it is when $\Delta G < 0$) then $\lambda_r < 0$, since by definition

$$\lambda_r = \frac{cov(\Delta s, r_0)}{V(r_0)}.$$

This proves part i).

To prove ii) notice that by virtue of the decomposition (S.2), if $0 \leq \Delta G < 2E(s_1\Delta r)$, then it follows that $E(r_0\Delta s) = cov(\Delta s, r_0) < 0$, and hence $\lambda_r < 0$.

By a similar logic, if $\Delta G \geq 2E(s_1\Delta r)$, then by (S.2) it follows that $cov(\Delta s, r_0) \geq 0$, and hence $\lambda_r \geq 0$. Furthermore, the above inequalities automatically guarantee that $\Delta G \geq 0$, as we know from the previous Lemma that $E(s_1\Delta r) \geq 0$. Together this establishes part iii) of the Proposition.

Finally, the impossibility in part iv) follows immediately from part i). □

Proof of Proposition 4. We begin by establishing a result that will be used at different points of the proof. In particular, notice that for any transformation of income $y = f(d)$, it follows that

$$\begin{aligned}\Delta E(ys) =& E(y_1 s_1) - E(y_0 s_0)\\ =& E(s_1 y_1) - E(s_1 y_0) + E(y_0 s_1) - E(y_0 s_0)\\ =& E(s_1 \Delta y) + E(y_0 \Delta s)\\ =& E(s_1 \Delta y) + cov(\Delta s, y_0),\end{aligned}$$

where the last equality follows because $cov(\Delta s, y_0) = E(y_0 \Delta s) - E(y_0)E(\Delta s)$ and $E(\Delta s) = 0$.

Rearranging the above equations we arrive at

$$cov(\Delta s, y_0) = \Delta E(ys) - E(s_1 \Delta y) \tag{S.3}$$

When $\alpha \neq 0, 1$, the GE index is

$$GE(\alpha \neq 0, 1) = \frac{E(s^\alpha - 1)}{\alpha(\alpha - 1)},$$

hence, for our particular transformation $y^e = s^{\alpha-1}/(\alpha - 1)$,

$$\begin{aligned}\Delta GE(\alpha \neq 0, 1) =& \frac{\Delta E(s^{\alpha-1} s)}{\alpha(\alpha - 1)}\\ =& \frac{\Delta E(s y^e)}{\alpha}.\end{aligned}$$

In the case of $\alpha = 1$,

$$GE(1) = E(s \log s),$$

hence, for $y^e = \log s$,

$$\begin{aligned}\Delta GE(1) =& \Delta E(s \log s)\\ =& \Delta E(s y^e).\end{aligned}$$

The above conditions can be expressed succinctly as

$$\Delta E(s y^e) = \alpha \Delta GE(\alpha \neq 0).$$

Substituting this last expression in (S.3), we have that

$$cov(\Delta s, y_0^e) = \alpha \Delta GE(\alpha \neq 0) - E(s_1 \Delta y^e). \tag{S.4}$$

Equation (S.4) will be useful in establishing several of the following results.

Proof of part i)

Our proof is established separately for different ranges of values of $\alpha \neq 0$.

Case $\alpha \neq 0, 1$

Notice first that the sign of λ_y is given by the sign of

$$\begin{aligned} cov(\Delta s, y_0^e) =& E(\Delta s \cdot y_0^e) - E(\Delta s)E(y_0^e) \\ =& E(\Delta s \cdot y_0^e) \\ =& E(\Delta s \cdot s_0^{\alpha-1})/(\alpha - 1). \end{aligned}$$

In other words, the condition for convergence when $\alpha \neq 0, 1$ is

$$\lambda_y < 0 \iff \frac{1}{(\alpha - 1)} \frac{1}{n} \sum (\Delta s \cdot s_0^{\alpha-1}) < 0. \tag{S.5}$$

Our proof of part i) for the case when $\alpha \neq 0, 1$ will make use of either equations (S.4) or (S.5).

Proof of part i) for $\alpha \in (0, 1)$

For this particular subcase when $\alpha \in (0, 1)$ we will use equation (S.4) together with the condition that $\Delta GE(\alpha) < 0$. In order to do so we need the following Lemma.[1]

Lemma. *If, for* $\alpha \in (0, 1)$, $\Delta GE(\alpha) < 0$, *then*

$$\sum s_1 \Delta s^{\alpha-1} < 0.$$

◁*Proof of Lemma*

Rewrite the LHS of this inequality as

$$\sum s_1 \Delta s^{\alpha-1} = \sum (s_1^\alpha - s_1 s_0^{\alpha-1}).$$

[1] This Lemma can be established following the results derived by Kumar and Choudhary (2011). We reproduce their proof applied to our context.

Let $(k, k') \in \mathbb{R}^2$, be conjugate, i.e.,

$$\frac{1}{k} + \frac{1}{k'} = 1.$$

Hardy *et al.* (1934) equation (2.8.4) show that for such (k, k') with $k < 1$ and for any $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n_+$, we can express the so-called Hölder's inequality as

$$\sum_i a_i b_i \geq \left(\sum_i a_i^k \right)^{\frac{1}{k}} \left(\sum_i b_i^{k'} \right)^{\frac{1}{k'}}. \tag{S.6}$$

Let $k = 1 - \alpha$, so $k' = \frac{\alpha - 1}{\alpha}$. Also let

$$a_i = s_{i1}^{\frac{\alpha}{1-\alpha}} \quad b_i = s_{i1}^{\frac{\alpha}{\alpha-1}} s_{i0}^{\alpha},$$

so that $a_i b_i = s_{i0}^{\alpha}$.

Substituting these terms into Hölder's inequality (S.6) (and dropping the individuals subscripts), we have

$$\sum s_0^{\alpha} \geq \left(\sum \left(s_1^{\frac{\alpha}{1-\alpha}} \right)^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \left(\sum \left(s_1^{\frac{\alpha}{\alpha-1}} s_0^{\alpha} \right)^{\frac{\alpha-1}{\alpha}} \right)^{\frac{\alpha}{\alpha-1}}.$$

The RHS of this equation is equal to

$$\left(\sum s_1^{\alpha} \right)^{\frac{1}{1-\alpha}} \left(\sum s_1 s_0^{\alpha-1} \right)^{\frac{\alpha}{\alpha-1}}.$$

Hence, we can re-arrange Hölder's inequality as

$$\left(\sum s_1 s_0^{\alpha-1} \right)^{\frac{\alpha}{1-\alpha}} \geq \left(\sum s_1^{\alpha} \right)^{\frac{1}{1-\alpha}} \left(\sum s_0^{\alpha} \right)^{-1}.$$

However, for $\alpha \in (0, 1)$, $\Delta GE(\alpha) < 0$ implies

$$\sum s_0^{\alpha} < \sum s_1^{\alpha}.$$

Hence,

$$\begin{aligned} \left(\sum s_1 s_0^{\alpha-1} \right)^{\frac{\alpha}{1-\alpha}} &\geq \left(\sum s_1^{\alpha} \right)^{\frac{1}{1-\alpha}} \left(\sum s_0^{\alpha} \right)^{-1} \\ &> \left(\sum s_1^{\alpha} \right)^{\frac{1}{1-\alpha}} \left(\sum s_1^{\alpha} \right)^{-1} \\ &= \left(\sum s_1^{\alpha} \right)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

Raising this last inequality to the power $\frac{1-\alpha}{\alpha}$ and rearranging terms establishes the Lemma. $\triangleright$

Having established that whenever $\alpha \in (0,1)$, a falling $GE(\alpha)$ implies

$$\sum s_1 \Delta s^{\alpha-1} < 0,$$

notice that this in turn implies that

$$E(s_1 \Delta y^e) > 0,$$

because $y^e = s^{\alpha-1}/\alpha - 1$ and $\alpha < 1$.

This last inequality together with equation (S.4)

$$cov(\Delta s, y_0^e) = \alpha \Delta GE(\alpha \neq 0) - E(s_1 \Delta y^e). \tag{S.4}$$

imply $cov(\Delta s, y_0^e) < 0$, since the covariance will be the sum of two negative terms. Since the sign of λ_y is given by this covariance, we have then established Proposition 4.i) for the case $\alpha \in (0,1)$.

Proof of part i) for $\alpha < 0$

To prove this result when $\alpha < 0$ we will directly establish the required sign in equation (S.5)

$$\lambda_y < 0 \iff \frac{1}{(\alpha-1)} \frac{1}{n} \sum (\Delta s \cdot s_0^{\alpha-1}) < 0. \tag{S.5}$$

Consider again Hölder's inequality (S.6),

$$\sum_i a_i b_i \geq \left(\sum_i a_i^k \right)^{\frac{1}{k}} \left(\sum_i b_i^{k'} \right)^{\frac{1}{k'}},$$

with $k^{-1} + k'^{-1} = 1$, such that $k' < 1$.

In this subcase let $k = \alpha$, so $k' = \frac{\alpha}{\alpha-1}$. Also let

$$a_i = s_{i1} \quad b_i = s_{i0}^{\alpha-1},$$

so that $a_i b_i = s_{i1} s_{i0}^{\alpha-1}$. Substituting these terms into (S.6) and rearranging leads to

$$\begin{aligned}\sum s_1 s_0^{\alpha-1} &\geq \left(\sum s_1^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum \left(s_0^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}} \\ &= \left(\sum s_1^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum s_0^{\alpha}\right)^{\frac{\alpha-1}{\alpha}} \\ &= \left(\sum s_0^{\alpha}\right) \left(\sum s_1^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum s_0^{\alpha}\right)^{-\frac{1}{\alpha}}. \qquad \text{(S.7)}\end{aligned}$$

Whenever $\alpha < 0$, a fall in $GE(\alpha)$ is equivalent to

$$\sum s_1^{\alpha} < \sum s_0^{\alpha}.$$

This last condition implies

$$\left(\sum s_1^{\alpha}\right)^{\frac{1}{\alpha}} > \left(\sum s_0^{\alpha}\right)^{\frac{1}{\alpha}},$$

because $\alpha < 0$. Equivalently,

$$\left(\sum s_1^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum s_0^{\alpha}\right)^{-\frac{1}{\alpha}} > 1.$$

Applying this inequality to equation (S.7) implies

$$\sum s_1 s_0^{\alpha-1} > \sum s_0^{\alpha},$$

or

$$\sum s_1 s_0^{\alpha-1} > \sum s_0 s_0^{\alpha-1}.$$

However, in the case where $\alpha < 0$, this last inequality is precisely the condition required for convergence $\lambda_y < 0$, as can be appreciated from equation (S.5)

$$\lambda_y < 0 \iff \frac{1}{(\alpha-1)} \frac{1}{n} \sum (\Delta s \cdot s_0^{\alpha-1}) < 0.$$

This proves Proposition 4.i) for the case $\alpha < 0$.

Proof of part i) for $\alpha > 1$

As in the previous subcase, to prove this result when $\alpha > 1$ we will directly establish the required sign for equation (S.5)

$$\lambda_y < 0 \iff \frac{1}{(\alpha-1)}\frac{1}{n}\sum(\Delta s \cdot s_0^{\alpha-1}) < 0. \tag{S.5}$$

Use again Hölder's inequality, this time in its canonical form (see Hardy *et al.*, 1934, eq. (2.8.3)). This version of the inequality states that if $(k, k') \in \mathbb{R}^2$ are conjugate, i.e., $k^{-1} + k'^{-1} = 1$, such that $k, k' > 1$, and for any $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$, then

$$\sum_i a_i b_i \leq \left(\sum_i a_i^k\right)^{\frac{1}{k}} \left(\sum_i b_i^{k'}\right)^{\frac{1}{k'}}. \tag{S.8}$$

Let $k = \alpha$, so $k' = \frac{\alpha}{\alpha-1}$. Also let

$$a_i = s_{i1} \quad b_i = s_{i0}^{\alpha-1}.$$

Substituting these terms into (S.8) (and dropping individuals subscripts), we have

$$\begin{aligned}\sum s_1 s_0^{\alpha-1} &\leq \left(\sum s_1^\alpha\right)^{\frac{1}{\alpha}} \left(\sum \left(s_0^{\alpha-1}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}} \\ &= \left(\sum s_1^\alpha\right)^{\frac{1}{\alpha}} \left(\sum s_0^\alpha\right)^{\frac{\alpha-1}{\alpha}}.\end{aligned} \tag{S.9}$$

When $\alpha > 1$, a fall in $GE(\alpha)$ is equivalent to

$$\sum s_1^\alpha < \sum s_0^\alpha.$$

Applying this inequality to (S.9) leads to

$$\sum s_1 s_0^{\alpha-1} < \left(\sum s_0^\alpha\right)^{\frac{1}{\alpha}} \left(\sum s_0^\alpha\right)^{\frac{\alpha-1}{\alpha}} = \sum s_0^\alpha.$$

We can rearrange this inequality as

$$\begin{aligned}\sum s_1 s_0^{\alpha-1} &< \sum s_0^\alpha, \\ \sum s_1 s_0^{\alpha-1} &< \sum s_0 s_0^{\alpha-1}, \\ \sum (\Delta s) s_0^{\alpha-1} &< 0.\end{aligned}$$

However, this condition is precisely the condition required for convergence $\lambda_y < 0$ in the case when $\alpha > 1$, as it can be seen from equation (S.5)

$$\lambda_y < 0 \iff \frac{1}{(\alpha-1)}\frac{1}{n}\sum(\Delta s \cdot s_0^{\alpha-1}) < 0. \tag{S.5}$$

This proves Proposition 4.i) for the case $\alpha > 1$.

Taken together, the three preceding results (for cases $\alpha \in (0,1)$, $\alpha < 0$, $\alpha > 1$) establish that Proposition 4.i) holds for $\alpha \neq 0, 1$. We next establish the remaining case, when $\alpha = 1$.

Proof of part i) for $\alpha = 1$

In this case we need to establish that the term $\sum s_1 \Delta \log s$ is always non-negative. However, this is a direct consequence of the Log-sum inequality well-known in the Information Theory literature (Han and Kobayashi, 2002).[2]

Since when $\alpha = 1$, we adopted $y^e = \log s$, then by the log-sum inequality, $E(s_1 \Delta y^e) \geq 0$, and this inequality together with equation (S.4)

$$cov(\Delta s, y_0^e) = \alpha \Delta GE(\alpha \neq 0) - E(s_1 \Delta y^e) \tag{S.4}$$

mean that, in this particular case, $cov(\Delta s, y_0^e) < 0$, since the covariance will be the sum of a negative term (ΔGE) a non-positive one ($-E(s_1 \Delta y^e)$). Since the sign of λ_y is given by this covariance, it means that Proposition 4.i) is true as well when $\alpha = 1$.

This completes the proof of Proposition 4.i) for all cases.

Proof of part ii)

Turn again to equation (S.4)

$$cov(\Delta s, y_0^e) = \alpha \Delta GE(\alpha \neq 0) - E(s_1 \Delta y^e). \tag{S.4}$$

From this equation it follows immediately that $cov(\Delta s, y_0^e) < 0$ (and thus $\lambda_y < 0$), as long as

$$\alpha \Delta GE(\alpha \neq 0) < E(s_1 \Delta y^e).$$

Hence part ii) of the proposition follows as long as $\Delta GE > 0$ and the above equation is satisfied.

[2] A proof is available from the authors upon request.

Proof of part iii)

This part of the proposition is similarly established. Since by equation (S.4) whenever

$$\alpha \Delta GE(\alpha \neq 0) \geq E(s_1 \Delta y^e).$$

and $\Delta GE(\alpha \neq 0) > 0$, there will be rising inequality and weak divergence, $\lambda_y \geq 0$.

Proof of part iv)

This impossibility follows immediately from part i) of the Proposition.

This completes the proof of all the cases in the Proposition. □

Proof of Proposition 5

Proof. When $\alpha = 0$, $GE(0) = E(\log(1/s))$, thus it follows that $\Delta GE(0) < 0$ if and only if $\sum \Delta \log s > 0$. Hence,

$$\Delta GE(0) < 0 \iff n^{-1} \sum (\log s_1 - \log s_0) > 0$$
$$\iff \sum \left(\log s_1^{\frac{1}{n}} - \log s_0^{\frac{1}{n}}\right) > 0 \iff \log\left(\prod s_1^{\frac{1}{n}}\right) > \log\left(\prod s_0^{\frac{1}{n}}\right)$$
$$\iff \prod s_1^{\frac{1}{n}} > \prod s_0^{\frac{1}{n}} \iff \prod \left(\frac{s_1}{s_0}\right)^{\frac{1}{n}} > 1.$$

The LHS of this last inequality is the geometric mean of s_1/s_0, i.e. $GM(s_1/s_0)$. Hence, $\Delta GE(0) < 0$ implies $GM(s_1/s_0) > 1$. The AM-GM inequality (see for instance, Hardy *et al.*, 1934, section 9, eq. (2.5.2)), applied to s_1/s_0 implies

$$E\left(\frac{s_1}{s_0}\right) \geq \prod \left(\frac{s_1}{s_0}\right)^{\frac{1}{n}}. \quad \text{(S.10)}$$

Equation (S.10) together with the condition $GM(s_1/s_0) > 1$ implies that

$$E\left(\frac{s_1 - s_0}{s_0}\right) > 0.$$

However, from Lemma S2, which shows that

$$sgn(\theta) = -sgn\left(E\left[\frac{s_1 - s_0}{s_0}\right]\right),$$

this last condition implies that $\theta < 0$, i.e. the exact proportional changes are convergent. This establishes part i).

From Lemma S2 we know that convergence in exact proportional changes requires

$$E\left(\frac{s_1 - s_0}{s_0}\right) > 0.$$

While $\Delta GE(0) > 0$ if and only if $E(\Delta \log s) < 0$.

Thus the condition in part ii) of the Proposition follows from joining these two conditions together. By the same logic, the result in part iii) is satisfied whenever

$$E\left(\frac{s_1 - s_0}{s_0}\right)$$

and $E(\Delta \log s)$ are both negative.

Finally, the impossibility in part iv) follows immediately from part i) of the Proposition.

□

Proof of Lemma 1. Proof of part ii)

Let s_0 be the initial vector of shares and let s_c be defined as in (1). Namely, let s_c be the vector of final-period shares sorted in ascending order of final income.

Theorem 2.1 in Fields and Fei (1978) implies that if the distribution of s_0 Lorenz-dominates that of s_c, i.e. if $LC_0 \succ LC_c$, then it is possible to go from s_0 to s_c by means of a sequence of rank-preserving disequalizing transfers.

One convenient way of representing such transfers is by indexing them as $h(i, j)$ where the first argument, i, indicates which individual is making a transfer and the second one, j, which one is receiving it.

Since the transfers are disequalizing, and no one makes a transfer to himself, they satisfy the following conditions:

$$h(i, j) = 0 \quad \text{for } d_{i0} \geq d_{j0}$$
$$h(i, j) \geq 0 \quad \text{for } d_{i0} < d_{j0} \quad \text{with strict inequality for some pair } \{i, j\}.$$

The total transfers made by individual i will be the sum over the second index j, namely

$$h(i,\cdot) = \sum_{j=1}^{n} h(i,j).$$

Similarly, the total transfers received by this same individual will be the sum over the first index, namely

$$h(\cdot,i) = \sum_{j=1}^{n} h(j,i).$$

Hence, the change in this person's income share can be expressed as the difference between the two previous quantities, i.e.

$$s_{ic} - s_{i0} = h(\cdot,i) - h(i,\cdot) = \sum_{j=1}^{n} h(j,i) - \sum_{j=1}^{n} h(i,j).$$

By construction, the sum of the share changes over all individuals is zero, hence each person's share loss is somebody else's share gain, and also each share gain is somebody else's loss. In other words, any given transfer $h(i,j)$ appears with a positive sign in the share change of individual j, and with a negative sign in the share change of individual i. Furthermore, at any given stage of the sequence of transfers, the sender i is always poorer than the receiver j, since the transfer is disequalizing. Hence, for each transfer $h(i,j)$ we have

$$h(i,j)(\tilde{s}_j - \tilde{s}_i) \geq 0,$$

where $\tilde{s}_i$ and $\tilde{s}_j$ are the shares of individuals i and j, respectively, at the given stage of the sequence of transfers where $h(i,j)$ takes place.

Since each of these transfers are rank-preserving, it follows that at any given stage of the sequence of transfers, $\tilde{s}_j - \tilde{s}_i \geq 0$ implies

$$h(i,j)(s_{j0} - s_{i0}) \geq 0.$$

Notice however, that SM in equation (10) can be rewritten as

$$\begin{aligned} SM &= n^{-1} \sum_i (s_{ic} - s_{i0}) s_{i0} \\ &= n^{-1} \sum_i \left(\sum_{j=1}^{n} h(j,i) - \sum_{j=1}^{n} h(i,j) \right) s_{i0}. \end{aligned}$$

That is, SM will be the average of terms $h(i,j)(s_{j0} - s_{i0})$ for all the transfers $h(i,j)$. Since these terms are non-negative, and some will be strictly positive, then SM will be positive.

In other words, we have shown that $LC_0 \succ LC_c$ implies $SM > 0$. However, by construction, the Lorenz curve of the vector s_c is the same as that of the final income vector s_1 (i.e. $LC_c = LC_1$), so we have that a Lorenz-worsening $LC_0 \succ LC_1$ implies $SM > 0$.

The proof of part i) follows by reproducing the previous steps, now with rank-preserving equalizing transfers. □

Proof of Lemma 2. Recall s_c is a permutation of s_1. In particular, s_c is sorted in ascending order of s_1, and we will assume that both s_0 and s_1 are sorted in ascending order of s_0. Since vectors s_1 and s_c have the same elements, the only differences between them are the ones due to positional changes. If nobody changes positions $s_c = s_1$, and $XM = 0$, trivially. Hence, we will assume from now on that $\exists\, i \leq n$ such that $s_{i1} \neq s_{ic}$.

Denote the difference between s_{i1} and s_{ic} by

$$\eta_i \equiv s_{i1} - s_{ic}.$$

In other words, η_i is the difference between the final-period share of the individual ranked i in the initial distribution and the final-period share of the individual ranked i in the final distribution, when each of these distributions is sorted in ascending order.

Also, denote the (ordered) set of individual indices by $I = \{1, 2, \ldots, n\}$.

Since we want to establish that

$$\sum_{i \in I} (s_{i1} - s_{ic}) s_{i0} \leq 0,$$

we need only include in the sum those individuals who changed position, since $\eta_i = 0$ for those who did not change position.

The (ordered) set of indices for individuals with non-zero positional change is denoted by $\tilde{I} = \{a(1), a(2), \ldots, a(m)\}$, for $m \leq n$. That is, there are m individuals who changed positions and $a(1), \ldots, a(m)$ are their indices in the original set I.

The next claim will be useful in what follows.

Claim. *For all $a(j) \in \tilde{I}$, let $a(m) \geq a(j)$. Then,*

$$\sum_{i=0}^{k} \eta_{a(m-i)} \leq 0 \quad \forall k < m.$$

$\triangleleft$ *Proof of Claim*

Start with $\eta_{a(m)} = s_{a(m)1} - s_{a(m)c}$. The share $s_{a(m)c}$ is the highest final-period share among those individuals who changed positions (since s_c is sorted in ascending order of final shares and $s_{a(m)c}$ is its last element). In contrast, $s_{a(m)1}$ is the final-period share of the initially richest individual who changed positions (since we assumed s_1 to be sorted in ascending order of initial income). Since the person initially in the position $a(m)$ must have moved lower in the distribution, it follows that $s_{a(m)1} < s_{a(m)c}$, and thus $\eta_{a(m)} < 0$.

Now consider the sum:

$$\eta_{a(m)} + \eta_{a(m-1)} = (s_{a(m)1} + s_{(m-1),1}) - (s_{a(m)c} + s_{a(m-1),c}).$$

In this expression, the terms in s_c are the two largest shares (among those who changed positions) because s_c is ordered in ascending order of s_1. In contrast, the terms $s_{a(m)1}$ and $s_{a(m-1),1}$ may or may not be the largest, hence

$$\eta_{a(m)} + \eta_{a(m-1)} \leq 0.$$

Now continue to the top three, top four, etc. The same logic as before yields

$$\sum_{i=0}^{k} \eta_{a(m-i)} = \sum_{i=0}^{k} s_{a(m-i),1} - \sum_{i=0}^{k} s_{a(m-i),c}.$$

Again, note that the elements in the s_c sum are the largest k+1 final shares among those who changed positions, while the elements in the s_1 sum need not be the largest k+1 final shares.

This establishes the claim.$\triangleright$

Since $\forall a(i) \in \tilde{I}$, $\eta_{a(i)} \neq 0$, we can partition the index set $\tilde{I}$ into alternating subsets of contiguous indices for individuals with positive and negative positional changes. That is, we can express

$$\tilde{I} = \{M_1, M_2, \ldots, M_h\} \quad h \leq m,$$

where the partition subsets M_k have the following properties:

i) For all $a(i) \in M_k$, it is either true that $\eta_{a(i)} > 0$ or $\eta_{a(i)} < 0$.

ii) For any sets M_k and M_l, with $k < l$, and for all $a(i) \in M_k$ and $a(j) \in M_l$, we have $a(i) < a(j)$.

Note that in the partition $\tilde{I} = \{M_1, M_2, \ldots, M_h\}$, the first subset, M_1, contains observations with positive η's. This is because $\eta_{a(1)} \in M_1$ and this term is strictly positive, since for all individuals who changed positions it is the case that $\eta_{a(j)} \neq 0$ and $s_{a(1)c}$ is the smallest final-period share among all position-changers. A similar logic establishes that the last subset in the partition, M_h, contains elements with negative η's.

To simplify notation we will denote the subsets with positive elements by P_j and the ones with negative changes by N_j. Hence, we can reexpress our partition as

$$\tilde{I} = \{P_1, N_1, \ldots, P_g, N_g\} \quad g < h,$$

Furthermore, for each of these P_j subsets denote their maximum as $\hat{p}_j = \max P_j$.

Next, define the following sums over such subsets:

$$SP_j = \sum_{i \in P_j} \eta_i; \quad SN_j = \sum_{i \in N_j} \eta_i; \quad S_j = SP_j + SN_j;$$
$$XP_j = \sum_{i \in P_j} \eta_i s_{i0}; \quad XN_j = \sum_{i \in N_j} \eta_i s_{i0}; \quad X_j = XP_j + XN_j.$$

For any X_j as defined above it is the case that:

$$\begin{aligned}
X_j &= \sum_{i \in N_j} \eta_i s_{i0} + \sum_{i \in P_j} \eta_i s_{i0} \\
&\leq \sum_{i \in N_j} \eta_i s_{i0} + s_{\hat{p}_j 0} \sum_{i \in P_j} \eta_i \quad (\text{since } \hat{p}_j = \max P_j,\ \eta_i > 0\ \ \forall i \in P_j, \\
&\qquad\qquad\qquad\qquad\qquad \text{and } s_0 \text{ is sorted in ascending order}) \\
&= \sum_{i \in N_j} \eta_i s_{i0} + s_{\hat{p}_j 0}(S_j - SN_j) \quad (\text{by definition of } S_j) \\
&= \sum_{i \in N_j} \eta_i (s_{i0} - s_{\hat{p}_j 0}) + s_{\hat{p}_j 0} S_j \quad (\text{since } SN_j = \sum_{i \in N_j} \eta_i).
\end{aligned}$$

Observe that $\forall i \in N_j$, it is the case that $\eta_i < 0$, and also $s_{i0} - s_{\hat{p}_j0} \geq 0$ (by the fact that s_0 is sorted in ascending order and for any subset j, if $i \in N_j$ and $l \in P_j$, then $l < i$). Therefore the following inequality holds

$$\sum_{i \in N_j} \eta_i (s_{i0} - s_{\hat{p}_j0}) \leq 0,$$

always.

Now consider the other term in the expression of X_j, namely, $s_{\hat{p}_j0} S_j$. Our goal is to prove that the summation of these terms across all subsets is also non-positive. To establish this result we will work with the partial sum of such terms, starting from the highest index g all the way down to an arbitrary $k < g$, that is

$$\sum_{h=0}^{k} s_{\hat{p}_{g-h}0} S_{g-h}.$$

In particular, it is the case that

$$\begin{aligned}
0 &\geq s_{\hat{p}_{g-k}0} \sum_{h=0}^{k} S_{g-h} \quad \text{(since } \sum_{h=0}^{k} S_{g-h} \leq 0 \text{ by the above Claim that} \\
&\qquad\qquad \sum_{i=0}^{k} \eta_{a(m-i)} \leq 0 \quad \forall k < m) \\
&= s_{\hat{p}_{g-k}0} \sum_{h=0}^{k-1} S_{g-h} + s_{\hat{p}_{g-k}0} S_{g-k} \\
&\geq s_{\hat{p}_{g-k+1}0} \sum_{h=0}^{k-1} S_{g-h} + s_{\hat{p}_{g-k}0} S_{g-k} \ \ \text{(as } \sum_{h=0}^{k-1} S_{g-h} \leq 0 \text{ and } s_{\hat{p}_{g-k}0} < s_{\hat{p}_{g-k+1}0}) \\
&= s_{\hat{p}_{g-k+1}0} \sum_{h=0}^{k-2} S_{g-h} + s_{\hat{p}_{g-k+1}0} S_{g-k+1} + s_{\hat{p}_{g-k}0} S_{g-k}.
\end{aligned}$$

Continuing these steps $k-2$ more times establishes that

$$\sum_{h=0}^{k} s_{\hat{p}_{g-h}0} S_{g-h} \leq 0,$$

as desired.

In other words we have shown that for any j

$$\sum_{i \in N_j} \eta_i (s_{i0} - s_{\hat{p}_j 0}) \leq 0,$$

and for any $k \leq g$,

$$\sum_{h=0}^{k} s_{\hat{p}_{g-h} 0} S_{g-h} \leq 0.$$

Hence, summing across all partitions we obtain

$$\sum_{j=1}^{g} X_j \leq \sum_{j=1}^{g} \left[\sum_{i \in N_j} \eta_i (s_{i0} - s_{\hat{p}_j 0}) + s_{\hat{p}_j 0} S_j \right] \leq 0,$$

but $n^{-1} \sum_{j=1}^{g} X_j = XM$, so this completes the proof of the Lemma. $\square$

Proof of Proposition 6. Consider the share-change regression

$$\Delta s \equiv s_1 - s_0 = \gamma_s + \delta_s s_0 + u_s.$$

The coefficient δ_s equals

$$\delta_s = \frac{cov(\Delta s, s_0)}{V(s_0)}.$$

Hence, its sign will be determined by the sign of the covariance

$$\begin{aligned} cov(\Delta s, s_0) &= n^{-1} \sum_i (s_{i1} - s_{i0}) s_{i0} - \overline{\Delta s} \cdot \overline{s_0} \\ &= n^{-1} \sum_i (s_{i1} - s_{i0}) s_{i0} \qquad (\text{since } \overline{\Delta s} = 0) \\ &= n^{-1} \sum_i [(s_{i1} - s_{ic}) + (s_{ic} - s_{i0})] s_{i0} \\ &= XM + SM \end{aligned}$$

for XM and SM defined in (10). Hence,

$$sgn(\delta_s) = sgn(XM + SM).$$

By Lemma 1.i), a Lorenz-improvement $LC_1 \succ LC_0 \implies SM < 0$. By Lemma 2, $XM \leq 0$ always. Hence, if $LC_1 \succ LC_0$ then $XM + SM < 0$, and therefore $\delta_s < 0$. This proves part i) of the Proposition.

Part ii) follows again from the equation $sgn(\delta_s) = sgn(XM + SM)$. Namely, if there is a Lorenz-worsening, $LC_1 \prec LC_0$, then by Lemma 1, $SM > 0$. However, if $|XM| > SM$, it will still be the case that $\delta_s < 0$.

If in contrast, $|XM| \leq SM$, then $SM + XM \geq 0$, and hence $\delta_s \geq 0$. This establishes part iii). Finally, part iv) immediately follows also from part i).

This completes the proof of the Proposition. □

Proof of Proposition 7. Let gm_0 denote the geometric mean of incomes at period 0, i.e.

$$gm_0 = exp\left(n^{-1}\sum_l \log d_{l0}\right).$$

Let $h > 0$ be a small rank-preserving transfer between two individuals i and j, with initial incomes satisfying $d_{i0} > d_{j0} > gm_0 * exp(1)$. Furthermore, assume that the only income change when going from period 0 to 1 is the transfer h between i and j, and all other incomes remaining unchanged.

It follows from Fields and Fei (1978) that if the transfer is equalizing it will lead to a Lorenz-improvement, and the opposite will occur if the transfer is disequalizing. This establishes the directions of the Lorenz-Dominance in parts a) and b) of the Proposition. The only result remaining to establish is the sign of the coefficient $\delta_{\log}$ in a log-change regression (11) under the stated income change.

Consider the case a) of a single rank-preserving *equalizing* transfer. That is the transfer goes from the richer person i to the poorer person j. Under the stated assumptions the sign of $\delta_{\log}$ will be determined by the covariance

$$cov(\Delta \log d, \log d_0) = n^{-1}\sum_l (\log d_{l1} - \log d_{l0}) \log d_{l0} - \overline{\Delta \log d} \cdot \overline{\log d_0}.$$

Note that all terms in the summation are zero except for $l \in \{i, j\}$, so we have

$$
\begin{aligned}
cov(\Delta \log d, \log d_0) &= n^{-1}\left[(\Delta \log d_i)\log d_{i0} + (\Delta \log d_j)\log d_{j0}\right] - \overline{\Delta \log d}\cdot\overline{\log d_0} \\
&= n^{-1}\left[(\log(d_{i0}-h) - \log d_{i0})\log d_{i0}\right] \\
&\quad + n^{-1}\left[(\log(d_{j0}+h) - \log d_{j0})\log d_{j0}\right] - \overline{\Delta \log d}\cdot\overline{\log d_0}
\end{aligned}
$$

First-order Taylor expansions of the first two terms around $h = 0$ are

$$
n^{-1}\left[(\log(d_{i0}-h) - \log d_{i0})\log d_{i0}\right] \cong -\frac{\log d_{i0}}{d_{i0}}\frac{h}{n}
$$

and

$$
n^{-1}\left[(\log(d_{j0}+h) - \log d_{j0})\log d_{j0}\right] \cong \frac{\log d_{j0}}{d_{j0}}\frac{h}{n}.
$$

A similar expansion for the average log-income change is

$$
\overline{\Delta \log d} \cong \frac{h}{n}\left(\frac{1}{d_{j0}} - \frac{1}{d_{i0}}\right).
$$

Hence, for a marginal transfer h,

$$
cov(\Delta \log d, \log d_0) \cong \frac{h}{n}\left(\frac{\log d_{j0} - \overline{\log d_0}}{d_{j0}} - \frac{\log d_{i0} - \overline{\log d_0}}{d_{i0}}\right).
$$

The sign of this covariance will be determined by the behavior of the function

$$
\frac{\log x - \overline{\log d_0}}{x}
$$

with derivative

$$
\frac{1 - \log x + \overline{\log d_0}}{x^2}.
$$

This derivative will be negative when

$$
x > exp(1) * gm_0.
$$

Hence, if individuals i, j have incomes $d_{i0} > d_{j0} > exp(1) * gm_0$, and an equalizing transfer is made from i to j, then

$$
\frac{\log d_{j0} - \overline{\log d_0}}{d_{j0}} - \frac{\log d_{i0} - \overline{\log d_0}}{d_{i0}}
$$

will have a positive sign, and so $\delta_{\log} > 0$. The case of a disequalizing transfer is proved similarly.

□

Proof of Lemma 3. Proof of Part i):

Recall that

$$PSM = \frac{1}{n} \sum_i \frac{s_{ic} - s_{i0}}{s_{i0}}.$$

Similar to what we did in the proof of Lemma 1, when there is a Lorenz-improvement we can go from s_0 to s_c through a sequence of rank-preserving equalizing transfers $h(j,i)$ from a richer individual j to a poorer one i. The share change for individual i when going from its initial income to s_c is

$$s_{ic} - s_{i0} = \sum_{j=1}^{n} h(j,i) - \sum_{j=1}^{n} h(i,j).$$

Hence, we can express PSM as

$$PSM = \frac{1}{n} \sum_i \frac{\sum_{j=1}^{n} h(j,i) - \sum_{j=1}^{n} h(i,j)}{s_{i0}}.$$

For any transfer $h(j,i)$ in such sequence it is the case that:

a) $h(j,i)$ contributes once with a positive sign to the share change of the receiver i and once with a negative sign to the share change of the sender j, and

b) at any given stage of the sequence of transfers, the sender j is always richer than the receiver i.

Hence, for each equalizing transfer $h(j,i)$, we have that the product

$$h(j,i) \left(\frac{1}{\tilde{s}_i} - \frac{1}{\tilde{s}_j} \right) \geq 0,$$

where $\tilde{s}_i$ and $\tilde{s}_j$ are the shares of individuals i and j, respectively, at the given stage of the sequence of transfers where $h(j,i)$ takes place.

Since each of these transfers are rank-preserving, it follows that at any given stage of the sequence of transfers, $\tilde{s}_j - \tilde{s}_i \geq 0$ implies $s_{j0} > s_{i0}$ and hence

$$h(j,i) \left(\frac{1}{s_{i0}} - \frac{1}{s_{j0}} \right) \geq 0.$$

The above implies that PSM will be the sum of terms $h(j,i)(\frac{1}{s_{i0}} - \frac{1}{s_{j0}})$ for all transfers $h(j,i)$. Since all these terms are non-negative, and some will be strictly positive, then PSM will be positive.

The proof of Part ii) is similar, now with rank-preserving disequalizing transfers.

□

Proof of Lemma 4. Recall s_c is a permutation of s_1. In particular, s_c is sorted in ascending order of s_1, and as in the proof of Lemma 2, we will assume that both s_0 and s_1 are sorted in ascending order of s_0. Since both vectors s_1 and s_c have the same elements, the only differences between them are the ones due to positional changes. If nobody changes positions $s_c = s_1$, and $PXM = 0$, trivially. Hence, we will assume from now on that $\exists\ i \leq n$ such that $s_{i1} \neq s_{ic}$.

Denote the difference between s_{i1} and s_{ic} by

$$\eta_i \equiv s_{i1} - s_{ic}.$$

In other words, η_i is the difference between the final-period share of the individual ranked i in the initial distribution and the final-period share of the individual ranked i in the final distribution, when each of these distributions is sorted in ascending order.

Also, denote the (ordered) set of individual indices by $I = \{1, 2, \ldots, n\}$.

Since we want to establish that

$$\sum_{i \in I} \frac{s_{i1} - s_{ic}}{s_{i0}} \geq 0,$$

we need only include in the sum those individuals who changed position, since $\eta_i = 0$ for those who did not change position.

The (ordered) set of indices for individuals with non-zero positional changes is denoted by $\tilde{I} = \{a(1), a(2), \ldots, a(m)\}$, for $m \leq n$. That is, there are m individuals who changed positions and $a(1), \ldots, a(m)$ are their indices in the original set I.

The next claim will be useful in what follows.

Claim.

$$\sum_{i=1}^{k} \eta_{a(i)} \geq 0 \quad \forall k \leq m.$$

It is worth noting that this result bears a close parallel to the Claim made in the proof of Lemma 2. The difference between these two Claims is that in the one from Lemma 2, the cumulative sum of η's was made by adding first the η's with the highest indices, while in this Claim the sum is made in the opposite direction, i.e., it starts with the η's corresponding to lowest indices in $\tilde{I}$.

◁*Proof of Claim:*

Note that $\eta_{a(1)} = s_{a(1)1} - s_{a(1)c} > 0$, since for all individuals who changed positions it is the case that $\eta_{a(j)} \neq 0$ and $s_{a(1)c}$ is the smallest final-period share among all position-changers. (In contrast, $s_{a(1)1}$ is the final-period share of the initially poorest individual who changed positions, and thus moved up in the distribution).

Now consider the sum:

$$\eta_{a(1)} + \eta_{a(2)} = (s_{a(1)1} + s_{a(2)1}) - (s_{a(1)c} + s_{a(2)c}).$$

The terms in s_c are the two smallest shares because s_c is ordered in ascending order of s_1. The terms in s_1 may or may not be the smallest, hence

$$\eta_{a(1)} + \eta_{a(2)} \geq 0.$$

Now continue to the bottom three, bottom four, etc. The same logic as before yields

$$\sum_{i=1}^{k} \eta_{a(i)} = \sum_{i=1}^{k} s_{a(i)1} - \sum_{i=1}^{k} s_{a(i)c}.$$

Again, note that the elements in the s_c sum are the smallest k final shares, while the elements in the s_1 sum need not be the smallest k final shares.

This establishes the claim.▷

As in the proof of Lemma 2, we can partition the index set $\tilde{I}$ into alternating subsets of contiguous indices for individuals with positive and negative positional changes. That is, we can express

$$\tilde{I} = \{P_1, N_1, \ldots, P_g, N_g\} \quad g \leq m.$$

Furthermore, for each of these N_j subsets denote their minimum as $\check{n}_j = \min N_j$.

Next, define the following sums over such subsets:

$$SP_j = \sum_{i \in P_j} \eta_i; \quad SN_j = \sum_{i \in N_j} \eta_i; \quad S_j = SP_j + SN_j;$$
$$PXP_j = \sum_{i \in P_j} \frac{\eta_i}{s_{i0}}; \quad PXN_j = \sum_{i \in N_j} \frac{\eta_i}{s_{i0}}; \quad PX_j = PXP_j + PXN_j.$$

For any PX_j as defined above, it is the case that:

$$\begin{aligned}
PX_j &= \sum_{i \in P_j} \frac{\eta_i}{s_{i0}} + \sum_{i \in N_j} \frac{\eta_i}{s_{i0}} \\
&\geq \sum_{i \in P_j} \frac{\eta_i}{s_{i0}} + \frac{1}{s_{\check{n}_j 0}} \sum_{i \in N_j} \eta_i \quad \text{(since } \check{n}_j = \min N_j, \quad \eta_i < 0 \ \ \forall i \in N_j \\
&\qquad\qquad\qquad\qquad\qquad \text{and } s_0 \text{ is sorted in ascending order)} \\
&= \sum_{i \in P_j} \frac{\eta_i}{s_{i0}} + \frac{1}{s_{\check{n}_j 0}} (S_j - SP_j) \quad \text{(by definition of } S_j) \\
&= \sum_{i \in P_j} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_j 0}} \right) + \frac{1}{s_{\check{n}_j 0}} S_j \quad \text{(since } SP_j = \sum_{i \in P_j} \eta_i).
\end{aligned}$$

For all $i \in P_j$, note that $\eta_i > 0$, and $s_{i0} \leq s_{\check{n}_j 0}$ (because s_0 is sorted in ascending order and for any subset j, if $i \in P_j$ and $l \in N_j$, then $i < l$). Therefore, the following inequality holds

$$\sum_{i \in P_j} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_j 0}} \right) \geq 0,$$

always.

Now consider the other term in the expression of PX_j, namely, $S_j / s_{\check{n}_j 0}$. Our goal is to prove that the summation of these terms across all subsets is also non-negative. To establish this result we will work with the partial sum of such terms starting from the lowest index 1 all the way up to an arbitrary $k < g$, i.e.

$$\sum_{h=1}^{k} \frac{1}{s_{\check{n}_h 0}} S_h.$$

In particular, it is the case that,

$$
\begin{aligned}
0 &\leq \frac{1}{s_{\check{n}_k 0}} \sum_{h=1}^{k} S_h \qquad \left(\text{since } \sum_{h=1}^{k} S_h \geq 0 \text{ by the above Claim that } \sum_{i=1}^{r} \eta_{a(i)} \geq 0 \quad \forall r \leq m\right) \\
&= \frac{1}{s_{\check{n}_k 0}} \sum_{h=1}^{k-1} S_h + \frac{1}{s_{\check{n}_k 0}} S_k \\
&\leq \frac{1}{s_{\check{n}_{k-1} 0}} \sum_{h=1}^{k-1} S_h + \frac{1}{s_{\check{n}_k 0}} S_k \quad \left(\text{as } \sum_{h=1}^{k-1} S_h \geq 0 \text{ and } s_{\check{n}_{k-1} 0} < s_{\check{n}_k 0}\right) \\
&= \frac{1}{s_{\check{n}_{k-1} 0}} \sum_{h=1}^{k-2} S_h + \frac{1}{s_{\check{n}_{k-1} 0}} S_{k-1} + \frac{1}{s_{\check{n}_k 0}} S_k.
\end{aligned}
$$

Continuing these steps $k-2$ more times establishes that

$$\sum_{h=1}^{k} \frac{1}{s_{\check{n}_h 0}} S_h \geq 0.$$

In other words, we have shown that for any j

$$\sum_{i \in P_j} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_j 0}} \right) \geq 0,$$

and for any $k \leq g$

$$\sum_{h=1}^{k} \frac{1}{s_{\check{n}_h 0}} S_h \geq 0.$$

Hence, summing across all partitions, we obtain

$$\sum_{j=1}^{g} PX_j \geq \sum_{j=1}^{g} \left[\sum_{i \in P_j} \eta_i \left(\frac{1}{s_{i0}} - \frac{1}{s_{\check{n}_j 0}} \right) + \frac{1}{s_{\check{n}_j 0}} S_j \right] \geq 0,$$

but $n^{-1} \sum_{j=1}^{g} PX_j = PXM$, so this completes the proof of the Lemma. □

Proof of Proposition 8. Note first that

$$\begin{aligned} E\left(\frac{\Delta s}{s_0}\right) &= \frac{1}{n}\sum_i \frac{s_{i1}-s_{i0}}{s_{i0}} \\ &= \frac{1}{n}\sum_i \frac{(s_{i1}-s_{ic})+(s_{ic}-s_{i0})}{s_{i0}} \\ &= PXM + PSM \end{aligned}$$

for PXM and PSM defined in equation (12).

By Lemma 3, a Lorenz-improvement implies $PSM > 0$, and by Lemma 4, $PXM \geq 0$ always. Hence, under a Lorenz-improvement the average percentage change in shares will be positive.

Now, we know from Lemma S2 that whenever average percentage changes in shares are positive, the exact proportional changes are convergent (and vice versa), i.e.

$$0 < E\left(\frac{s_1 - s_0}{s_0}\right) \iff \theta < 0.$$

This establishes part i) of the Proposition.

If there is a Lorenz-worsening, then $PSM < 0$ by Lemma 3. Furthermore, if $PXM > |PSM|$ then the sum $PXM + PSM$, and thus the average percentage change in shares will be positive. As we saw in the above paragraph, this implies that the exact proportional changes are convergent. This establishes part ii).

Similarly, if there is a Lorenz-worsening (and therefore $PSM < 0$) and if $PXM \leq |PSM|$, then the sum $PXM + PSM$, and thus the average percentage change in shares will be less or equal to 0. Again by Lemma S2, this implies that the exact proportional changes are weakly divergent, establishing part iii) of the Proposition.

Finally, the impossibility in part iv) follows immediately from part i). □

Proof of Proposition 9. Proof of part i):

By Proposition 6.i) a Lorenz-improvement ($LC_1 \succ LC_0$) implies share convergence, $\delta_s < 0$, or $\beta_s < 1$. Coupling this with Lemma S3, which establishes that $\beta_d = \beta_s(1+g)$, it follows that whenever $g \leq 0$, a Lorenz-improvement implies $\beta_d = \beta_s(1+g) < 1$, i.e. convergence in dollars.

Part ii) is the contrapositive of part i) under a scenario of negative growth, i.e. $g \leq 0$. Similarly, the impossibility result of part iii) follows immediately from part i).

Finally, part iv) follows from Lemma S3 and the fact that it is possible to have income transitions satisfying the remaining combinations of Lorenz-worsening and improvements, convergent/divergent panel dollar changes, and positive/negative growth. See Table S3 for specific examples. □

S.3 Additional Results for Selected Inequality Indices

S.3.1 Coefficient of Variation

Proposition S1. ***Changes in the Coefficient of Variation and Convergence in Shares***
Let CV denote the coefficient of variation of income in dollars. Let δ_s be defined by the share-change regression

$$\Delta s = \gamma_s + \delta_s s_0 + u_s.$$

Then:

i) Falling CV and convergent share-changes
If the CV falls, then the regression of share-changes on initial shares is convergent, i.e., $\delta_s < 0$.

ii) Rising CV and convergent share-changes
If the CV rises and if $\Delta CV^2 < V(\Delta s)$, then the regression of share-changes on initial shares is convergent, i.e., $\delta_s < 0$.

iii) Rising CV and divergent share-changes
If the CV rises and if $\Delta CV^2 \geq V(\Delta s)$, then the regression of share-changes on initial shares is weakly divergent, i.e., $\delta_s \geq 0$.

iv) Falling CV and divergent share-changes
It is impossible to simultaneously have a falling CV, i.e., $\Delta CV < 0$, and weakly divergent share-changes, i.e., $\delta_s \geq 0$.

Corollary 1. *If the regression of share-changes on initial shares is weakly divergent, i.e., $\delta_s \geq 0$, then the CV must rise.*

Proof of Proposition S1. The proposition follows immediately by applying Proposition 2 in the main text to income shares s, and by remembering that $CV^2(d) = V(s)$ by Lemma S1. □

S.3.2 Atkinson Family of Indices

In Propositions S2 and S3 (and their corollaries), we present results for the Atkinson family of inequality indices equivalent to those in Propositions 4 and 5 for the Generalized Entropy family.

Proposition S2. ***Changes in the Atkinson Index with parameter $\epsilon \neq 0$ and Convergence of Share-Changes-on-y***

Let $A(\epsilon \neq 0)$ denote the Atkinson measure with parameter $\epsilon \neq 0$, which may take any real value less than one, $\epsilon < 1$, i.e.

$$A(\epsilon \neq 0) = 1 - \left[n^{-1} \sum_{i=1}^{n} s_{it}^{\epsilon} \right]^{\frac{1}{\epsilon}}$$

Let y^a ("a" for "Atkinson") be a monotonically increasing transformation of income defined by

$$y^a = \frac{s^{\epsilon - 1}}{\epsilon - 1}.$$

Let λ_y be defined by the share-change-on-y regression

$$\Delta s = \kappa_y + \lambda_y y_0^a + e_y.$$

Finally, let

$$\tau(A) = \frac{(1 - A)^{\epsilon}}{\epsilon(\epsilon - 1)}$$

be a monotonically increasing transformation of $A \equiv A(\epsilon \neq 0)$.

Then:

i) Falling $A(\epsilon \neq 0)$ and convergent share-changes-on-y

If the Atkinson index, $A(\epsilon \neq 0)$, falls, then the share-changes-on-y regression is convergent, i.e., $\lambda_y < 0$.

ii) Rising $A(\epsilon \neq 0)$ and convergent share-changes-on-y

If the Atkinson index, $A(\epsilon \neq 0)$, rises, and if $\epsilon \Delta \tau(A) < E(s_1 \Delta y^a)$, then the share-changes-on-y regression is convergent, i.e., $\lambda_y < 0$.

iii) Rising $A(\epsilon \neq 0)$ and divergent share-changes-on-y

If the Atkinson index, $A(\epsilon \neq 0)$, rises, and if $\epsilon \Delta \tau(A) \geq E(s_1 \Delta y^a)$, then the share-changes-on-y regression is weakly divergent, i.e., $\lambda_y \geq 0$.

iv) Falling $A(\epsilon \neq 0)$ and divergent share-changes-on-y

It is impossible to simultaneously have a falling Atkinson index, i.e., $\Delta A(\epsilon \neq 0) < 0$, and weakly divergent share-changes-on-y, i.e., $\lambda_y \geq 0$.

Proof of Proposition S2

Proof. Parts i) and iv)

It is easy to see that for the Atkinson index

$$\Delta A(\epsilon \neq 0) \gtreqless 0 \iff -\epsilon \sum \Delta s^{\epsilon} \gtreqless 0.$$

Similarly, for the Generalized Entropy indices with parameter $\alpha \neq 0$,

$$\Delta GE(\alpha \neq 0) \gtreqless 0 \iff \alpha(\alpha - 1) \sum \Delta s^{\alpha} \gtreqless 0.$$

Thus, whenever $\epsilon = \alpha < 1$ the changes in these indices will have the same sign, i.e. whenever $\epsilon = \alpha < 1$ and $\epsilon \neq 0$,

$$sgn(\Delta A(\epsilon \neq 0)) = sgn(\Delta GE(\alpha \neq 0)).$$

Also, when $\epsilon = \alpha < 1$ and $\epsilon \neq 0$ our monotonic transformation of income y^a equals the corresponding transformation y^e used in the share-changes-on-y regressions of Proposition 4, for the Generalized Entropy index.

Hence, the parts i) and iv) of the current Proposition follow by setting $\epsilon = \alpha < 1$ in parts i) and iv) of Proposition 4.

Parts ii) and iii)

Let $\tau(A)$ be a monotonically increasing transformation of the Atkinson measure $A = A(\epsilon \neq 0) = 1 - (E(s^{\epsilon}))^{1/\epsilon}$, defined by

$$\begin{aligned} \tau(A) =& \frac{(1 - A)^{\epsilon}}{\epsilon(\epsilon - 1)} \\ =& \frac{E(s^{\epsilon})}{\epsilon(\epsilon - 1)}. \end{aligned}$$

For y^a defined as

$$y^a = \frac{s^{\epsilon - 1}}{\epsilon - 1}$$

we can thus express

$$\tau(A) = \frac{E(s^{\epsilon})}{\epsilon(\epsilon - 1)}$$
$$= \frac{E(sy^{a})}{\epsilon}.$$

Hence,

$$\epsilon\Delta\tau(A) = \Delta E(sy^{a}).$$

However, as was shown in the proof of Proposition 4, for any transformation of income $y = f(d)$, the sign of λ_y in the share-changes-on-y regression is determined by

$$cov(\Delta s, y_0) = \Delta E(ys) - E(s_1 \Delta y). \tag{S.3}$$

Substituting the term $\Delta E(sy^{a})$ for this particular case, we have that there is convergence of share-changes-on-y, i.e., $\lambda_y < 0$, whenever

$$cov(\Delta s, y_0^{a}) = \epsilon\Delta\tau(A) - E(s_1 \Delta y^{a}) < 0.$$

A rise in $A(\epsilon \neq 0)$ together with the above inequality establish case ii) of the Proposition.

Similarly, a rise in $A(\epsilon \neq 0)$ and

$$\epsilon\Delta\tau(A) \geq E(s_1 \Delta y^{a})$$

establish the case iii) in the Proposition.

This completes the proof of all the cases in the Proposition.

□

Corollary 2. *If the share-changes-on-y regression is weakly divergent, i.e., $\lambda_y \geq 0$, then the Atkinson index $A(\epsilon \neq 0)$ must weakly rise.*

Proposition S3. ***Changes in the Atkinson Index with parameter $\epsilon = 0$ and Convergence of Exact Proportional Changes***

Let $A(0)$ denote the Atkinson index with parameter $\epsilon = 0$, i.e.

$$A(0) = 1 - \prod_{i=1}^{n} s_{it}^{1/n}$$

Let θ be defined by the exact proportional changes regression

$$(d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}. \tag{5}$$

Then:

i) Falling $A(0)$ and convergent exact proportional changes

If the Atkinson index with parameter $\epsilon = 0$, $A(0)$, falls, then the exact proportional changes regression is convergent, i.e., $\theta < 0$.

ii) Rising $A(0)$ and convergent exact proportional changes

If

$$E(\Delta \log s) < 0 < E\left(\frac{s_1 - s_0}{s_0}\right),$$

then the Atkinson index with parameter $\epsilon = 0$, $A(0)$, rises, and the exact proportional changes regression is convergent, i.e., $\theta < 0$.

iii) Rising $A(0)$ and divergent exact proportional changes

If both $E(\Delta \log s)$ and $E\left(\frac{s_1 - s_0}{s_0}\right)$ are negative, then the Atkinson index with parameter $\epsilon = 0$, $A(0)$, rises, and the exact proportional changes regression is divergent, i.e., $\theta > 0$.

iv) Falling $A(0)$ and divergent exact proportional changes

It is impossible to simultaneously have a falling Atkinson index with parameter $\epsilon = 0$, i.e., $\Delta A(0) < 0$, and a weakly divergent exact proportional changes regression, i.e., $\theta \geq 0$.

Proof of Proposition S3

Proof. In this case it is easy to establish that

$$\Delta A(0) \gtreqless 0 \iff -\sum \Delta \log s \gtreqless 0,$$

which happens to be the same condition for $\Delta GE(0) \gtreqless 0$. It therefore follows that whenever $\epsilon = \alpha = 0$ the changes in the Atkinson and the Generalized Entropy indices will have the same sign, i.e.,

$$sgn(\Delta A(0)) = sgn(\Delta GE(0)).$$

Thus this Proposition follows from Proposition 5. □

Corollary 3. *If the exact proportional changes regression is weakly divergent, i.e., $\theta \geq 0$, then the Atkinson index with parameter $\epsilon = 0$, $A(0)$, must weakly rise.*

The results presented in section 3.2 in the main text and in this subsection establish a link between each of the commonly used inequality indices and a specified panel income change regression. In Table S1 we summarize which indices pair-up with which regressions.

S.4 Extensions to Cases Involving Single Lorenz Crossings

We define first the property of "transfer sensitivity" and define a class of inequality indices satisfying this property.

Definition S1. ***Transfer-sensitive Inequality Measures*** *(Shorrocks and Foster, 1987)*
An inequality measure $I(d)$ is transfer sensitive (TS) iff $I(d_1) < I(d_0)$ whenever d_1 is obtained from d_0 by a series of transfers whereby at each stage i) a progressive transfer occurs at lower income levels, ii) a regressive transfer occurs at higher income levels, iii) ranks remain unchanged, and iv) the variance of incomes remains unchanged.

In this section, we will present a result for the following class of inequality measures.

Table S1: Relative Inequality Measures and Income Change Regressions

Measure	Income-Change Regression
Coefficient of Variation	$\Delta s = \gamma_s + \delta_s s_0 + u_s$
	$d_1 = \alpha_d + \beta_d d_0 + u_d$
Variance of Logarithms	$\Delta \log d = \gamma_{log} + \delta_{log} \log d_0 + u_{log}$
Gini	$\Delta s = \kappa_r + \lambda_r r_0 + e_r$
Atkinson	
$\epsilon < 1, \epsilon \neq 0$	$\Delta s = \kappa_y + \lambda_y s_0^{\epsilon-1}/(\epsilon - 1) + e_y$
$\epsilon = 0$	$(d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}$
Theil's first measure	$\Delta s = \kappa_{\log s} + \lambda_{\log s} \log s_0 + e_{\log s}$
Generalized Entropy	
$\alpha \neq 0, 1$	$\Delta s = \kappa_y + \lambda_y s_0^{\alpha-1}/(\alpha - 1) + e_y$
$\alpha = 1$	$\Delta s = \kappa_{\log s} + \lambda_{\log s} \log s_0 + e_{\log s}$
$\alpha = 0$	$(d_1 - d_0)/d_0 = \phi + \theta d_0 + u_{pch}$

s denotes income in dollars (d) as a share of mean income, $\mu = E(d)$ i.e. $s = d/\mu$. r is the normalized income rank running from 1/n to 1.

Definition S2. I_{TS} ***Class of Inequality Measures***
Let $I_{TS}(d)$ *be the class of inequality measures satisfying transfer sensitivity (TS), the transfer principle (T), scale-independence (S), population-independence (P), and anonymity (A).*

We also formally define a single crossing from above.

Definition S3. ***Single Lorenz Crossing From Above***

Denote by $LC(d;p)$ *the Lorenz curve ordinate corresponding to the lowest 100p% of income recipients, for* $p \in [0,1]$*. The Lorenz curve for a distribution d is said to intersect that of* d' once from above *iff there exists* $p^* \in (0,1)$ *and intervals* $P \equiv [0,p^*]$ *and* $P' \equiv [p^*,1]$ *such that*

$$LC(d;p) \geq LC(d';p) \quad \forall p \in P \quad \text{and } > \text{ for some } p \in P$$
$$LC(d;p) \leq LC(d';p) \quad \forall p \in P' \quad \text{and } < \text{ for some } p \in P'.$$

Intuitively, again, this entails equalizing transfers toward persons at the low end and disequalizing transfers toward persons at the high end of the distribution.

With these definitions, we can now state a result relating the I_{TS} class of inequality indices to the share change regression (9).

Proposition S4. ***Changes in Transfer Sensitive Inequality Indices under a Single Lorenz-Crossing from Above, and Convergence in Shares***

Let δ_s *be defined by the regression*

$$\Delta s = \gamma_s + \delta_s s_0 + u_s. \tag{9}$$

If the Lorenz curve of d_1 *intersects that of* d_0 *once from above and* $CV(d_1) \leq CV(d_0)$*, then:*

i) for all measures in the $I_{TS}(d)$ *class,* $I_{TS}(d_1) < I_{TS}(d_0)$*, and*

ii) $\delta_s < 0$.

Proof of Proposition S4

Proof. Part i) is derived in Shorrocks and Foster (1987), Corollary 1.

Also, part i) of Proposition S1 states that $\delta_s < 0$, whenever $CV_1 < CV_0$.

It only remains to show that if $CV_1 = CV_0$ it will be the case that $\delta_s < 0$. In order establish this, transform regression (9) to its final-on-initial form,

$$s_1 = \gamma_s + (\delta_s + 1)s_0 + u_s.$$

Since $cov(s_0, u_s) = 0$ by construction, taking the variance of both sides leads to

$$V(s_1) = (\delta_s + 1)^2 V(s_0) + V(u_s).$$

Thus, we can rewrite the change in the variance of s as

$$\Delta V(s) = V(s_1) - V(s_0) = \delta_s(\delta_s + 2)V(s_0) + V(u_s).$$

Since by Lemma S1, $CV^2 = V(s)$, if $\Delta CV = 0$, this means that

$$0 = \delta_s(\delta_s + 2)V(s_0) + V(u_s),$$

which in turn implies that $\delta_s \leq 0$.

If $V(u_s) > 0$, it must be the case that $\delta_s < 0$. If $V(u_s) = 0$, then $u_i = 0$ for all individuals and either $\delta_s = 0$ or $\delta_s = -2$. Consider the first case, namely $V(u_s) = 0$ and $\delta_s = 0$, then the regression would be

$$\begin{aligned} \Delta s_i &= \gamma_s + \delta_s s_{i0} + u_i \\ \Delta s_i &= \gamma_s. \end{aligned}$$

However, since $E(\Delta s) = 0$, this implies that $\gamma_s = 0$. In other words, if $V(u_s) = 0$ and $\delta_s = 0$, then $s_{i1} = s_{i0}$ for all individuals. This is a contradiction, since we assumed a changing Lorenz curve across periods. Hence, in our context, the assumption $V(u_s) = 0$ is only compatible with $\delta_s = -2 < 0$. Hence, this proves that in all cases $\delta_s < 0$. □

Intuitively, Proposition S4 states that when the Lorenz curve of final incomes crosses that of initial incomes from above and the coefficient of variation falls or remains constant, then transfer sensitive inequality indices fall, and share changes are convergent. However, it bears mentioning that in this context share-convergence arises because of the changes in the CV, and not because of the particular type of Lorenz-crossing.

To empirically illustrate Proposition S4, consider the transition

$$d_0 = [1, 5, 10, 11] \rightarrow d_1 = [2, 4, 9, 12].$$

In this case the conditions of the Proposition are satisfied, namely there is: i) a single-crossing from above in the Lorenz curves, and ii) a falling CV (from 0.596 to 0.587). In this case it is readily verified that commonly used indices that are Transfer-Sensitive like the Atkinson family and Generalized Entropy with $\alpha < 2$ will mark a reduction in inequality, and there is share convergence as well.

S.5 Summary of Special Cases

In this section we establish the findings described in section 3.5 in the main text. These findings are presented in summarized in Table S2. To establish the results we will need to refer to some of the equations developed in the text. To facilitate the reading of the following proofs, we reproduce next the key equations used in establishing the results.

Structural and Exchange Mobility

$$\begin{aligned} SM &= \frac{\sum_i (s_{ic} - s_{i0}) s_{i0}}{n} \\ XM &= \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n}. \end{aligned} \tag{S.11}$$

for s_c defined in Definition 1 in the main text.

Structural and Exchange Mobility and Share Changes

$$sgn(\delta_s) = sgn(XM + SM). \tag{S.12}$$

(established in the proof of Proposition 6)

Proportional Structural and Exchange Mobility

$$\begin{aligned} PSM &= \frac{1}{n} \sum_i \frac{s_{ic} - s_{i0}}{s_{i0}} \\ PXM &= \frac{1}{n} \sum_i \frac{s_{i1} - s_{ic}}{s_{i0}}. \end{aligned} \tag{S.13}$$

Proportional Structural and Exchange Mobility and Proportional Changes

$$sgn(\theta) = -sgn(PXM + PSM). \tag{S.14}$$

(established in the proof of Proposition 8)

Table S2: Convergence/Divergence of Panel Income Changes under Special Cases

	Special Case 1: Lorenz Curves Unchanged, Positional Changes Only	**Special Case 2: Lorenz Curves Unchanged, Uniform Proportional Income Change**	**Special Case 3: Lorenz Worsening, No Positional Changes**
Share Changes	Convergent	Neutral	Divergent
Dollar Changes	Convergent	Convergent if $g < 0$ Divergent if $g > 0$	Divergent if $g \geq 0$ otherwise ambiguous
Proportional Changes			
Log Dollar Approx.	Convergent	Neutral	Ambiguous
Exact Prop. Changes	Convergent	Neutral	Divergent

Convergence/Divergence of shares, dollars and log-dollars changes is assessed by $\Delta y = \gamma_y + \delta_y y_0 + u_y$, where y is income measured in shares, dollars and in log-dollars, respectively. Convergence/Divergence in exact proportional changes is assessed by the regression $(d_1 - d_0)/d_0 = \phi + \theta d_0 + e_{pch}$.

Proof of Table S2. **Special Case 1:** Positional Change Only.

Share Changes

When only positional change takes place, the structural mobility term SM in equation (S.11) is equal to zero. Hence, by equation (S.12) the sign of δ_s equals the sign of XM, which in this case of positional changes will be negative by Lemma 2 in the main text.

Dollar Changes

The sign of δ_d can be established through Lemma S3,

$$\beta_d = \beta_s(1 + g).$$

Since we established that in this case $\delta_s < 0$, then it follows that $\beta_s < 1$. Furthermore since $g = 0$, it must be that $\beta_d < 1$, or equivalently, $\delta_d < 0$.

Log-Dollar Changes

In the case of the log-dollar regression, the sign of δ_{log} is determined by

$$cov(\Delta \ln d, \ln d_0) = n^{-1} \sum_i (\ln d_{i1} - \ln d_{i0}) \ln d_{i0} - \overline{\Delta \ln d} \cdot \overline{\ln d_0}.$$

Since the anonymous distributions are unchanged, then $\overline{\Delta \ln d} = 0$, and the covariance is simply

$$cov(\Delta \ln d, \ln d_0) = n^{-1} \sum_i (\ln d_{i1} - \ln d_{i0}) \ln d_{i0}.$$

In Lemma 2 we established that

$$\sum_i (s_{i1} - s_{ic}) s_{i0}$$

is negative when positional rearrangements occur. Following similar steps, we can establish that in our context with only positional rearrangements, it will be the case that

$$\sum_i (\ln d_{i1} - \ln d_{i0}) \ln d_{i0} < 0.$$

This implies $cov(\Delta \ln d, \ln d_0) < 0$, and therefore $\delta_{log} < 0$.

Exact Proportional Changes

In the case of exact proportional changes, under the stated assumptions, PSM in equation (S.13) equals zero, and PXM will be strictly positive by Lemma 4. By equation (S.14), this implies $\theta < 0$.

Special Case 2: Uniform Proportional Change.

Share Changes

A uniform proportional change entails $d_1 = \kappa d_0$ for some $\kappa > 0$. It is obvious that all shares will remain unchanged and so $\delta_s = 0$, and therefore $\beta_s = 1$.

Dollar Changes

In the case of the dollar change regression we know from Lemma S2 that

$$\beta_d = \beta_s(1 + g).$$

Since $\beta_s = 1$, there will be convergence ($\beta_d < 1$) or divergence ($\beta_d > 1$) in dollar changes depending on whether g is smaller or greater than zero.

Log-Dollar Changes

Log-dollar changes will be

$$\ln d_1 - \ln d_0 = \ln \kappa,$$

i.e. log-dollar changes are constant for all individuals, and thus $\delta_{log} = 0$.

Exact Proportional Changes

Exact proportional changes become

$$\frac{d_1 - d_0}{d_0} = \frac{d_0(\kappa - 1)}{d_0} = \kappa - 1,$$

hence again $\theta = 0$.

Special Case 3: Lorenz Worsening and no positional change.

Share Changes

In the absence of positional changes, the s_1 and s_c vectors in equation (1) in the main text are the same. Therefore, in (S.11),

$$XM = \frac{\sum_i (s_{i1} - s_{ic}) s_{i0}}{n}$$

is equal to zero. By equation (S.12) it follows that $sgn(\delta_s) = sgn(SM)$. Lemma 1.ii) in the main text tells us that a Lorenz-worsening implies $SM > 0$, which in turn implies share-divergence in Special Case 3, i.e. $\delta_s > 0$, or equivalently $\beta_s > 1$.

Dollar Changes

In the case of the dollar change regression we know from Lemma S3 that

$$\beta_d = \beta_s(1 + g),$$

hence, if $\beta_s > 1$ (as it is under this scenario), there will be divergence in dollars ($\beta_d > 1$) as long as $g \geq 0$. For $g < 0$, β_d can take values smaller or greater than 1 depending on which term (β_s or $1 + g$) dominates.

Log-Dollar Changes

The ambiguity of log-dollar changes in Special Case 3 can be established by producing examples with opposite signs for δ_{log}.

Consider for instance the transition

$$[1, 1, 1, 1, 1, 1, 1, 1, 7, 8] \rightarrow [1, 1, 1, 1, 1, 1, 1, 1, 6, 9].$$

In this case there is a Lorenz worsening without positional changes, yet the log-change regression registers convergence ($\beta_{log} = 0.99$). However, the transition

$$[5, 20] \rightarrow [5, 25]$$

is a Lorenz worsening without positional changes and divergence in log-dollars ($\beta_{log} = 1.16$).

Exact Proportional Changes

Again in the absence of positional changes, s_1 and s_c are the same and hence PXM

$$PXM = \frac{1}{n} \sum_i \frac{s_{i1} - s_{ic}}{s_{i0}}$$

is zero.

Lemma 3 established that a Lorenz-worsening implies $PSM < 0$. By equation (S.14) we know that

$$sgn(\theta) = -sgn(PSM + PXM).$$

Hence, under the assumptions of this scenario, $\theta > 0$. □

S.6 A Matrix of Possibilities

We have identified three ways of determining the direction of change in relative inequality - i) Lorenz-improvement and Lorenz-worsening, ii) Change in a Lorenz-consistent relative inequality index, and iii) Change in Lorenz-inconsistent relative inequality measures - and four ways of assessing divergence or convergence: i) Share changes, ii) Dollar changes, iii) Log-dollar changes, and iv) Exact proportional changes.

Can each possible combination of rising or falling relative inequality and divergent or convergent panel income changes arise? We show in this section that the answer is yes, provided they are measured suitably. Table S3 displays examples of each of the possible combinations.

To demonstrate the possibilities of most of the combinations, just two people are needed. But to get the remaining combinations, we need to complicate the examples by adding more people and choosing our measures carefully.

This matrix of possibilities shows a number of other things:

- Many but not all possibilities involve a Lorenz-dominance relationship (27 out of 32 cells to be precise). When such a result holds, it is stronger than a result for a particular inequality index, because it holds for *all* Lorenz-consistent inequality indices.

- All four convergence rows are consistent with Lorenz-worsening and Lorenz-improvement, both in times of positive economic growth and in times of negative economic growth.

- Falling relative inequality as gauged by Lorenz-improvement is consistent with all four types of convergence but not with divergent share changes or divergent exact proportional changes, regardless of whether growth is positive or negative, nor with divergent dollar changes in times of negative economic growth.[3]

- Falling relative inequality as gauged by a suitably chosen Lorenz-consistent inequality measure is consistent with all types of divergence. However, some of these combinations can only arise when the two Lorenz curves cross in a particular way.

- Suppose that all individuals were to remain in the same positions within the income distribution. Even in the absence of positional change, it is possible to have Lorenz-worsening together with convergent dollar changes in periods of economic decline. It is also possible, in the absence of positional change, to have Lorenz-worsening together with convergent log-dollar changes, under both positive and negative growth.[4]

[3] These inconsistencies are established in section 3.

[4] It is easy to generate examples in these cells using only two individuals who change positions between periods.

Table S3: Matrix of Possibilities in Times of Economic Growth and Decline.

Final on Initial Regression: $y_1 = \alpha_y + \beta_y y_0 + u_y$
Changes Regression: $\Delta y_1 = \gamma_y + \delta_y y_0 + u_y$
Exact Proportional Changes Regression: $\frac{d_1 - d_0}{d_0} = \phi + \theta d_0 + u_{pch}$

		Economic Growth Positive		**Economic Growth Negative**	
		Falling Relative Inequality	**Rising Relative Inequality**	**Falling Relative Inequality**	**Rising Relative Inequality**
Convergence/Divergence	**Convergent**				
	Share changes ($\beta_s < 1 \iff \delta_s < 0$)	[5,20]→[10,20]^{LD}	[5,20]→[25,5]^{LD}	[5,25]→[5,20]^{LD}	[7,23]→[20,5]^{LD}
	Dollar changes ($\beta_d < 1 \iff \delta_d < 0$)	[5,20]→[10,20]^{LD}	[5,20]→[25,5]^{LD}	[5,25]→[5,20]^{LD}	[7,23]→[5,20]^{LD}
	Proportional changes *Log-dollar Approx.* ($\beta_{\log} < 1 \iff \delta_{\log} < 0$)	[5,20]→[10,20]^{LD}	[1,1,1,1,1,1,1,1,6.1,8.89]→ [1,1,1,1,1,1,1,1,6,9]^{LD}	[5,25]→[5,20]^{LD}	[1.1,407,418]→ [1,360,390]^{LD}
	Exact Prop. changes ($\theta < 0$)	[5,20]→[10,20]^{LD}	[5,20]→[25,5]^{LD}	[5,25]→[5,20]^{LD}	[7,23]→[20,5]^{LD}
	Divergent				
	Share changes ($\beta_s > 1 \iff \delta_s > 0$)	[1,5,10]→ [2,4,25]*	[5,20]→[5,25]^{LD}	[60,320,1000]→ [54,150,876]*	[10,20]→[5,20]^{LD}
	Dollar changes ($\beta_d > 1 \iff \delta_d > 0$)	[5,20]→[7,23]^{LD}	[5,20]→[5,25]^{LD}	[20,90,180]→ [20,61,180]*	[10,20]→[5,20]^{LD}
	Proportional changes *Log-dollar Approx.* ($\beta_{\log} > 1 \iff \delta_{\log} > 0$)	[1,360,390]→ [1.1,407,418]^{LD}	[5,20]→[5,25]^{LD}	[1,1,1,1,1,1,1,1,6,9]→ [1,1,1,1,1,1,1,1,6.1,8.89]^{LD}	[10,20]→[5,20]^{LD}
	Exact Prop. changes ($\theta > 0$)	[1,5,10]→ [2,4,25]*	[5,20]→[5,25]^{LD}	[60,320,1000]→ [54,150,876]*	[10,20]→[5,20]^{LD}

Notes: LD: Lorenz-Dominance
*: Lorenz-dominance is not possible in this cell. Inequality can be judged to have fallen using the income share of the poorest tercile.

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