CCP Estimation of Dynamic Discrete Choice Models with Unobserved Heterogeneity*

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Abstract

Standard methods for solving dynamic discrete choice models involve calculating the value function either through backwards recursion (finite-time) or through the use of a fixed point algorithm (infinite-time). Conditional choice probability (CCP) estimators provide a computationally cheaper alternative but are perceived to be limited both by distributional assumptions and by being unable to incorporate unobserved heterogeneity via finite mixture distributions. We extend the classes of CCP estimators that need only a small number of CCP's for estimation. We also show that not only can finite mixture distributions be used in conjunction with CCP estimation, but, because of the computational simplicity of the estimator, an individual's location in unobserved space can transition over time. Monte Carlo results suggest that the algorithms developed are computationally cheap with little loss in precision.

Keywords: dynamic discrete choice, unobserved heterogeneity

1 Introduction

Standard methods for solving dynamic discrete choice models involve calculating the value function either through backwards recursion (finite-time) or through the use of a fixed point algorithm (infinite-time). Conditional choice probability (CCP) estimators provide an alternative to these techniques which involves mapping the value functions into the probabilities of making particular

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decisions. While CCP estimators are much easier to compute than estimators based on obtaining the full solution and have experienced a resurgence in the literature on dynamic games,[1] there are at least two reasons why researchers have been reticent to employ them in practice. First, it is perceived that the mapping between CCP's and value functions is simple only in specialized cases. Second, it is believed that CCP estimators cannot be adapted to handle unobserved heterogeneity.[2] This latter criticism is particularly damning as one of the fundamental issues in labor economics, and indeed one of the main purposes of structural microeconomics, is the explicit modelling of selection.

We show that, for a wide class of Generalized Extreme Value (GEV) distributions of the error structure, the value function depends only on the one period ahead CCP's and, in single-agent problem, often depends upon only the one period ahead CCP's for a single choice. The class of problems we discuss is quite large and includes dynamic games where one of the decisions is whether to exit. Further, unobserved heterogeneity via finite mixture distributions is not only easily incorporated into the algorithm, but the finite mixture distributions can transition over time as well. Previous work on incorporating unobserved heterogeneity has been restricted to cases of permanent unobserved heterogeneity in large part because of the computational burdens associated with allowing for persistence, but not permanence, in the unobserved heterogeneity distribution. Using insights from the macroeconomics literature on regime switching and the computational simplicity of CCP estimation, we show that incorporating persistent but time-varying heterogeneity comes at very little computational cost.

We adapt the EM algorithm, and in particular its application to sequential likelihood developed in Arcidiacono and Jones (2003), to two classes of CCP estimators that between them cover a wide class of dynamic optimization problems and sequential games with incomplete information, based on representations developed in Hotz et al (1994) and Altug and Miller (1998). Our techniques can be also readily applied to models with discrete and continuous choices by exploiting the Euler equation representation given in Altug and Miller (1998).

The algorithm begins by making a guess as to the CCP's for each unobserved type, conditional on the observables. Given this initial guess, we iterate on two steps. First, given the type-specific CCP's, maximize the pseudo-likelihood function with the unobserved heterogeneity integrated out.

[1] See Aguirregabiria and Mira (2006), Bajari, Benkard, and Levin (2006), Pakes, Ostrovsky, and Berry (2004), and Pesendorfer and Schmidt-Dengler (2003).

[2] A third reason is that to perform policy experiments it is often necessary to solve the full model. While this is true, using CCP estimators would only involve solving the full model once for each policy simulation as opposed to multiple times in a maximization algorithm.

Second, update the type-specific CCP's using the parameter estimates. These updates can take two forms. In stationary models, the updates can come from the likelihoods themselves. In models with finite time horizons, while the updates can come from the likelihoods themselves, a second method may be computationally cheaper. Namely, similar to the EM algorithm, we calculate the probability that an individual is a particular type. These conditional type probabilities are then used as weights in forming the updated CCP's from the data.

We illustrate the small sample properties of our estimator using two sets of Monte Carlos designed to highlight the two methods of updating the type-specific CCP's. The first is a finite horizon model of teen drug use and schooling decisions. Students in each period decide to whether to stay in school and, if the choice is to stay, use drugs. Before using drugs, individuals only have a prior as to how well they will enjoy the experience. However, upon using drugs, students discover their drug 'type' and this is used in informing their future decisions. Here we illustrate both ways of updating the CCP's, using the likelihoods or the conditional probabilities of being particular a particular type as weights. Results of the Monte Carlo show that both methods of updating the CCP's yield estimates similar to that of full information maximum likelihood with little loss in precision.

The second is a dynamic entry/exit example with unobserved heterogeneity in the demand levels for particular markets which in turn affects the values of entry and exit. Here the unobserved heterogeneity is allowed to transition over time and the example explicitly incorporates dynamic selection. The type-specific CCP's are updated using the likelihoods evaluated at the current parameter estimates and the current type-specific CCP's. The results suggest that incorporating time-varying unobserved heterogeneity is not only feasible but computationally simple and yields precise estimates even of the transitions on the unobserved state variables.

Our work is most closely related to the nested pseudo-liklelihood estimators developed by Aguirreggabiria and Mira (2006), and Buchinsky, Hahn and Hotz (2005). Both papers seek to incorporate a fixed effect within their CCP estimation framework drawn from a finite mixture. Aguirregabiria and Mira (2006) show how to incorporate unobserved characteristics of markets in dynamic games, where the unobserved heterogeneity only affects the utility function itself. In contrast our analysis demonstrates how to incorporate unobserved heterogeneity into both the utility functions and the transition functions and are thereby account for the role of unobserved heterogeneity in dynamic selection. Buchinsky, Hahn and Hotz (2005) use the tools of cluster analysis, seeking conditions on the model structure that allow them to identify the unobserved type of each agent, whereas we only identify the distribution of unobserved heterogeneity across agents. Thus their approach seems most applicable in models where there are relatively small numbers of long lived agents which may or may not be comparable to each other, whereas our approach is applicable to large populations where the focus is on the unobserved proportions that partition it.

2 The Framework

We consider a dynamic programming problem in which an individual makes a sequence of discrete choices d_t over his lifetime $t \in \{1, \ldots, T\}$ for some $T \leq \infty$. The choice set has the same cardinality K at each date t, so we define d_t by the multiple indicator function $d_t = (d_{1t}, \ldots, d_{Kt})$ where $d_{kt} \in \{0, 1\}$ for each $k \in \{1, \ldots, K\}$ and

$$\sum\nolimits_{k=1}^{K} d_{kt} = 1$$

A vector of characteristics (s_t, ε_t) fully describes the individual at each time t, where s_t are a set of time-varying characteristics, and $\varepsilon_t \equiv (\varepsilon_{1t}, \ldots, \varepsilon_{Kt})$ is independently and identically distributed over time with continuous support and distribution function $G(\varepsilon_t)$. The vector s_t evolves as a Markov process, depending stochastically on the choices of the individual. We model the transition from s_t to s_{t+1} conditional on the choice $k \in \{1, \ldots, K\}$ with the probability distribution function $F_k(s_{t+1} | s_t)$. We assume the current utility of an individual with characteristics (s_t, ε_t) from choosing alternative k by setting $d_{kt} = 1$, is additively separable in s_t and ε_t , and can be expressed as $u_k(s_t) + \varepsilon_{kt}$. The individual sequentially observes (s_t, ε_t) and maximizes the expected discounted sum of utilities

$$E\left\{\sum\nolimits_{t=1}^{T} \sum\nolimits_{k=1}^{K} \beta^t d_{kt} \left[u_k(s_t) + \varepsilon_{kt}\right]\right\}$$

where $\beta \in (0, 1)$ denotes the fixed geometric discount factor.

Let $d_t^o \equiv (d_{1t}^o, \ldots, d_{Kt}^o)$ denote the optimal decision rule for this problem where $d_{kt}^o \equiv d_{kt}(s_t, \varepsilon_t)$ for each $k \in \{1, \ldots, K\}$, define the conditional valuation functions by

$$v_k(s_t) = E\left\{\sum\nolimits_{t=1}^{T} \sum\nolimits_{k=1}^{K} \beta^t d_{kt}^o \left[u_k(s_t) + \varepsilon_{kt}\right]\right\}$$

and the conditional choice probabilities as

$$\xi_k(s_t) = \int d_{kt}(s_t, \varepsilon_t) \, dG(\varepsilon_t)$$

The representation theorem of Hotz and Miller (1993) implies there is a mapping from the conditional choice probabilities to the conditional valuation functions, which we now denote as

$$\psi[\xi(s_t)] = \begin{bmatrix} \psi_2[\xi(s_t)] \\ \vdots \\ \psi_K[\xi(s_t)] \end{bmatrix} = \begin{bmatrix} v_2(s_t) - v_1(s_t) \\ \vdots \\ v_K(s_t) - v_1(s_t) \end{bmatrix}$$

The expected contribution of the ε_{kt} disturbance to current utility, conditional on the state s_t, is found by integrating over the region in which the k^{th} action is taken, so appealing to the representation theorem

$$\begin{aligned} & \int \left[d_{kt}\left(s_t, \varepsilon_t\right) \varepsilon_{kt} \right] dG\left(\varepsilon_t\right) \\ = & \int 1\left\{ \varepsilon_{kt} - \varepsilon_{jt} \geq u_j\left(z_t\right) - u_k\left(s_t\right) + v_j\left(s_t\right) - v_k\left(s_t\right) \text{ for all } j \in \{1, \ldots, J\} \right\} \left[d_{kt}\left(s_t, \varepsilon_t\right) \varepsilon_{kt} \right] dG\left(\varepsilon_t\right) \\ = & \int 1\left\{ \varepsilon_{kt} - \varepsilon_{jt} \geq u_j\left(s_t\right) - u_k\left(s_t\right) + \psi_j\left[\xi\left(s_t\right)\right] - \psi_k\left[\xi\left(s_t\right)\right] \text{ for all } j \in \{1, \ldots, J\} \right\} \left[d_{kt}\left(s_t, \varepsilon_t\right) \varepsilon_{kt} \right] dG\left(\varepsilon_t\right) \\ \equiv & \; w_k\left[\xi\left(s_t\right)\right] \end{aligned}$$

It now follows that the conditional valuation functions can be expressed as a mapping of the conditional choice probabilities for states that might be visited in the future

$$v_k\left(s_t\right) = E\left\{ \sum\nolimits_{t=1}^{T} \sum\nolimits_{k=1}^{K} \beta^t \xi_k\left(s_t\right)\left[u_k\left(s_t\right) + w_k\left[\xi\left(s_t\right)\right]\right] \right\}$$

3 Finite Dependence

We specialize the model by assuming that optimal decision making is not affected by any event that occurred in the distant past providing certain choice sequences were taken in the intervening periods, a concept called finite dependence. Introduced by Altug and Miller (1998) to faciliate estimation of their labor supply model of human capital, this concept can be extended to include a broad class of models in labor economics and industrial organization, including the terminal state dynamic programming models of Hotz and Miller (1993), renewal models considered by Rust (1987), and incomplete information games of entry and exit of Pakes, Ostrovosky and Berry (2004).

In particular, suppose after ρ periods a particular state can be reached through a series of choices given any initial choice. In this case, the difference in expected future utility across any two choices can be expressed as a function of the ρ period ahead flow utilities and CCP's. We illustrate the finite dependence property with a set of examples that highlight cases where only one-period ahead CCP's are necessary to calculate the expected future utility differences. These cases have a renewal action that is present in, for example, the Rust bus-engine problem and virtually all entry/exit games. The last example illustrates how finite dependence can still be used even if no renewal action is present. We then formally define the set of problems defined by finite dependence.

3.1 Example 1: Renewal problems

We first focus on a simple two choice model. Given the framework described above, Rust (1987) shows that if we make the additional assumption that the ϵ_t's are i.i.d. extreme value we can write the v's as:

$$v_0(s_t, \epsilon_t) = u_0(s_t, \epsilon_t) + \beta \left(\int \ln \left(\sum_{k=0}^{1} \exp(v_k(s_{t+1})) \right) f(s_{t+1}|s_t, d_t = 0) + \gamma \right)$$

$$v_1(s_t, \epsilon_t) = u_1(s_t, \epsilon_t) + \beta \left(\int \ln \left(\sum_{k=0}^{1} \exp(v_k(s_{t+1})) \right) f(s_{t+1}|s_t, d_t = 1) + \gamma \right)$$

where γ is Euler's constant.

Note that what is inside the log is the denominator for the probability of choosing any of the alternatives. This will hold true for any representation where the ϵ's come from a GEV distribution. This implies that the v's can be rewritten as:

$$v_0(s_t, \epsilon_t) = u_0(s_t, \epsilon_t) + \beta \left(\int [v_0(s_{t+1}) - \ln(\xi_0(s_{t+1}))] f(s_{t+1}|s_t, d_t = 0) + \gamma \right)$$

$$v_1(s_t, \epsilon_t) = u_1(s_t, \epsilon_t) + \beta \left(\int [v_0(s_{t+1}) - \ln(\xi_0(s_{t+1}))] f(s_{t+1}|s_t, d_t = 1) + \gamma \right)$$

There are some cases when the v_0's are particularly easy to calculate. For example, if $d_t = 0$ is an absorbing state, v_0 is often specified as something that is linear in the parameters. If it is possible to consistently estimate the transitions $f(s_{t+1}|s_t, d_t)$ and the one period ahead p_0's, then the obtaining estimates of the $u_1(z_t)$'s simplifies to estimating a linear in the parameters logit. Note that whether T is finite or infinite has no bearing on the problem.

The condition that the probability of choosing an alternative depends only on current utility and on a one period ahead choice probability of a single choice is broader than just the cases satisfying the terminal state property. To see this, note that the probability of choosing an alternative is always relative to one of the choices:

$$\begin{aligned} v_1(s_t, \epsilon_t) - v_0(s_t, \epsilon_t) &= u_1(s_t, \epsilon_t) - u_0(s_t, \epsilon_t) \\ &+ \beta \int [\ln(\xi_0(s_{t+1}))][f(z_{t+1}|z_t, d_t = 0) - f(s_{t+1}|s_t, d_t = 1)] \\ &+ \beta \int [v_0(s_{t+1})][f(s_{t+1}|s_t, d_t = 1) - f(s_{t+1}|s_t, d_t = 0)] \end{aligned}$$

If v_0 does not depend the lagged choice except through the one period ahead flow utility then the future component effectively cancels out leaving the $v(s_t)$'s once again linear in the utility parameters. This is a special case of Altug and Miller (1998), which establishes how to write CCP's when there is finite time dependence. Note that this does not rule out transitions of the state variables provided the transitions do not depend upon the choice. For ease of exposition, we label options where the value function depends only on the lagged choice through the current utility as having the one-period time dependence (OPTD) property. Many problems of interest naturally exhibit this property. Rust's bus engine problem is one, along with virtually every game that involves an exit decision.

This renewal property, where the difference in the value functions depends only on the linear flow utility, the one-period ahead transitions on the state variables, and the one-period ahead choice probabilities of a single choice, is not restricted solely to logit errors. Consider the case where the errors follow a nested logit specification and the 'outside option' depends upon the current state only through the one period ahead flow utlity, that is, it satisfies the OPTD property. The conditional value function is then:

$$\begin{aligned} v_j(z_t, \epsilon_t) &= u_j(z_t, \epsilon_t) + \beta\gamma \\ &+\beta \int \ln\left(\left(\sum_{k=1}^{K} \exp\left(\frac{v_k(s_{t+1})}{\sigma}\right)\right)^{\sigma} + \exp(v_0(s_{t+1}))\right) f(s_{t+1}|s_t, d_{tj}) \end{aligned}$$

Writing the future component in terms of the probability of choosing the outside good yields:

$$\begin{aligned} v_j(s_t, \epsilon_t) &= u_j(s_t, \epsilon_t) + \beta\gamma \\ &+\beta \int [v_0(s_{t+1}) - \ln(\xi_0(s_{t+1}))] f(s_{t+1}|s_t, d_{tj}) \end{aligned}$$

Note that this expression is identical to the simple logit example above. Differencing with respect to v_0 then yields the same cancellations as before.

Conditional on one of the options having the OPTD property, the expression above will be the same for a broad class of correlation structures. Consider a mapping $G : R^{K+1} \to R^1$ that satisfies the restrictions given in McFadden (1978) such that:

$$F(\epsilon_0, \epsilon_1, ..., \epsilon_K) = \exp\left(-G(e^{\epsilon_0}, e^{\epsilon_1}, ..., e^{\epsilon_K})\right)$$

is the cdf of multivariate extreme value distribution. Whenever $G(\cdot)$ can be written as:

$$G(e^v) = G(e^{v_1(s_t)}, ..., e^{v_K(s_t)}) + e^{v_0(s_t)}$$

the conditional value function for choosing k can be written as:

$$v_k(s_t, \epsilon_t) = u_k(s_t, \epsilon_t) + \beta \int \ln\left(G(e^{v_1(s_t)}, ..., e^{v_K(s_t)}) + e^{v_0(s_t)}\right) f(s_{t+1}|s_t, d_{tk})$$

Differencing with respect to v_0 and substituting in for the future value term with $[\exp(v_0(z_{t+1})]/\xi_0$ yields:

$$\begin{aligned} v_k(z_t, \epsilon_t) - v_0(s_t, \epsilon_t) &= u_k(s_t, \epsilon_t) - u_0(s_t, \epsilon_t) + \beta \int [\ln(\xi_0(s_{t+1}))][f(s_{t+1}|s_t, d_{t0}) - f(s_{t+1}|s_t, d_{tk})] \\ &\quad + \beta \int [v_0(s_{t+1})][f(s_{t+1}|s_t, d_{tk}) - f(s_{t+1}|s_t, d_{t0})] \end{aligned}$$

If choice 0 is a terminal state or if v_0 does not depend upon the choice made in the previous period except in the flow utility then the last line is easily calculated since the future portion cancels out. Note that this structure can accommodate quite complex error structures. For example, Bresnahan, Stern, and Trajtenberg (1997) allow errors to be correlated across multiple nests. Hence, only the CCP's for the outside good are needed to avoid solving the backwards-recursion or fixed point problem. Moreover, by using the techniques developed below to incorporate unobserved heterogeneity, the GEV assumption is quite weak. In particular, we can approximate correlation among the ϵ's with a finite mixture distribution and keep the additive separability of the outside option conditional on the draw from the mixture distribution.

3.2 Example 2: Dynamic Entry/Exit

To further illustrate the simplicity and versatility of our estimator, we show how it can be applied to the dynamic entry/exit game that forms the basis for the second Monte Carlo discussed in section **??**. The game is in discrete time with an infinite horizon. Players are assumed to employ strategies consistent with a Markov Perfect Equilibrium. There are M markets with one potential entrant arriving in each market in each period. Potential entrants choose whether or not to enter the market and, once in the market, continue to choose whether or not to stay. If the firm leaves the market, it disappears, which also occurs if it chooses not to enter (whether the firm dies or enters the pool of potential entrants to be assigned to a market is not relevant to the problem for large M).For simplicity, we assume that there can be at most 2 firms in a market.

Firms are identical but subject to private i.i.d. profit shocks for staying in or entering a market. These shocks are distributed i.i.d. Type I extreme value and only affect the cost of staying in the market. Since we are focusing on a simple entry/exit example, we assume that firm identity is not important, except for whether the firm is an incumbent or a potential entrant (and only the number of such firms matters). All choices occur simultaneously. Expected lifetime profits for firm i depend upon:

1. Whether there is another player, $E_j \in \{0, 1\}$

2. Whether firm i is an incumbent, $I_i \in \{0, 1\}$

3. If there is another player, whether that player is an incumbent, $I_j \in \{0, 1\}$

4. A state variable $s \in \{0, 1\}$ which transitions according to $f(s'|s)$.

The realized flow profits for i, denoted π, then depend upon whether the firm ends up a monopolist or a duopolist, $D \in \{0, 1\}$, whether the firm started out as an incumbent, and the state of demand. We can then express expected lifetime profits from entering as:

$$v_i(E_j, I_i, I_j, s) = \sum_{k=0}^{1} \xi_k(1, I_j, I_i, s)\left(\pi(k, I_i, s) + \beta \sum_{s'=1}^{2} V(1, 1, k, s')f(s'|s)\right) + \epsilon_i$$

where ξ_0 and ξ_1 are the probabilities of the other firm staying out of and in the market, respectively. With the profits for exiting left as a deterministic scrap value SV, the i.i.d. Type I extreme value ϵ's imply that we can write the previous expression as:

$$v_i(E_j, I_i, I_j, s) = \sum_{k=0}^{1} \xi_k(1, I_j, I_i, s)\left(\pi(k, I_i, s) + \beta\gamma + \beta SV - \beta \sum_{s'=1}^{2} \ln[\xi_0(1, 1, k, s')]f(s'|s)\right) + \epsilon_i$$

3.3 Example 3: Female Labor Supply

We now show an example where the future value terms depend upon more than the one-period CCP's. Each period until T a woman chooses whether or not to work. Earnings at her work depend upon her experience, h_t, and whether or not she worked in the previous period. Let $d_t = 1$ if the woman works and 0 if she does not. Since the v's will depend upon accumulated experience, leaving the workforce does not amount to starting over. For both the choice to work at time t and the choice to not work at time t we need to write the expected future utility for each of the choices such that the v term factored out from the expected future utility for both choices ρ periods ahead is the same and hence cancel out. Let the ϵ's associated with each of the choices be iid Type I extreme value. The value function for working is:

$$v_1(h_t, d_{t-1}, \epsilon_t) = u_1(h_t, d_{t-1}, \epsilon_t) + \ln\left[\exp(v_1(h_t + 1, 1)) + \exp(v_0(h_t + 1, 1))\right]$$

Since eventually we are going to be working with $v_1(h_t, d_{t-1}) - v_0(h_t, d_{t-1})$ we want to choose to reformulate the expected future utility above such that a different formulation for the expected future utility for not working will lead to a cancelation. Since by working an additional year of work experience is accumulated, we want to reformulate the expected future utility conditional on working as the conditional of value of not working plus a function of the CCP's:

$$\begin{aligned} v_1(h_t, d_{t-1}, \epsilon_t) &= u_1(h_t, d_{t-1}, \epsilon_t) + v_0(h_t+1, 1) + \ln\left[\exp(v_1(h_t+1) - (v_0(h_t+1,1)) + 1\right] \\ &= u_1(h_t, d_{t-1}, \epsilon_t) + v_0(h_t+1, 1) - \ln(\xi_0(h_t+1, 1)) \end{aligned}$$

Substituting in for $v_0(h_t + 1, 1)$ with the flow utility plus the expected future utility and again differencing out with respect to the no work option yields:

$$v_1(h_t, d_{t-1}, \epsilon_t) = u_1(h_t, d_{t-1}, \epsilon_t) - \beta \ln(\xi_0(h_t+1, 1)) + \beta u_0(h_t+1, 1) - \beta^2 \ln(\xi_0(h_t+1, 0)) + \beta^2 v_0(h_t+1, 0)$$

If we can then write down $v_0(h_t, d_{t-1}, \epsilon_t)$ such that the last term is also $\beta^2 v_0(h_t + 1, 0)$ then once we take differences these terms will cancel out. We will then have the differenced conditional value functions written solely as a function of the two-period ahead flow utilities and the two-period ahead conditional choice probabilities. For v_0, we want to now write the one-period ahead expected future utility relative to working and write the two period-ahead expected future utility relative to not working yielding:

$$v_0(h_t, d_{t-1}, \epsilon_t) = u_0(h_t, d_{t-1}, \epsilon_t) - \beta \ln(\xi_1(h_t, 1)) + \beta u_1(h_t, 0) - \beta^2 \ln(\xi_0(h_t+1, 1)) + \beta^2 v_0(h_t+1, 0)$$

Now taking $v_1(h_t, d_{t-1}) - v_0(h_t, d_{t-1})$ yields the cancelation:

$$\begin{aligned} v_1 - v_0 &= u_1(h_t, d_{t-1}, \epsilon_t) - u_0(h_t, d_{t-1}, \epsilon_t) + \beta \Big[u_0(h_t+1, 1) - u_1(h_t, 0, s_t) - \ln(\xi_0(h_t+1, 1,)) + \ln(\xi_1(h_t, 0)) \\ &\quad + \beta\Big(u_0(h_t+1, 0) - u_0(h_{t+1}, 1) - \ln(\xi_0(h_t+1, 0) + \ln(\xi_0(h_t+1, 1)\Big)\Big] \end{aligned}$$

3.4 The Finite Dependence Class

To define the full finite dependence class we introduce some extra notation. Let $S \times S$ denote the Cartesian product of pairs of states (r, s). Loosely speaking, finite dependence means returning, in a finite number of periods, to the S diagonal states generically denoted by $(s, s) \in R_0$, from any coordinate pair $(r, s) \in S \times S$ that can be reached from R_0 within a single period.

First consider the set of (r, s) coordinates that might occur after one period displacements from R_0, reached by any choice pair $(j, k) \in D \times D$ that determines the probability support of one period outcomes according to the transitions $F_j(r\,|s_0)$ and $F_k(s\,|s_0)$. More formally this set, denoted by $A \subseteq S \times S$, is defined as

$$A \equiv \{(r, s) \in S \times S : F_j(r\,|s_0)\, F_k(s\,|s_0) > 0 \text{ for some } (j, k) \in D \times D \text{ and } s_0 \in S\}$$

and by construction is symmetric; if $(r,s)\in A$, then $(s,r)\in A$ too.

Now let R_1 denote the set of (r,s) coordinate pairs that can reach R_0 in one period for sure given some (typically non-optimal but perhaps optimal) choice pair and the state transition matrices. That is

$$R_1 \equiv \{(r,s)\in S\times S : F_j\left(s_0\,|r\right)F_k\left(s_0\,|s\right)=1 \text{ for some } (j,k)\in D\times D \text{ and } s_0\in S\}$$

By definition, associated with every coordinate pair $(r,s)\in R_1$ there are two actions, j and k, that respectively take state r to state s_0 and state s to state s_0, meaning $F_j\left(s'\,|r\right)=F_k\left(s'\,|s\right)=0$ for all $s'\neq s_0$.

We can recursively define R_h as the set of paired states that are moved closer together by some action pair (j,k), in the sense of reaching a lower indexed set R_g for some $g<h$ the very next period.

$$R_h \equiv \left\{\begin{array}{r} (r,s)\in S\times S : \left[\sum_{r'=1}^{S}\sum_{s'=1}^{S} 1\left\{(r',s')\in \bigcup_{g=1}^{h-1} R_g\right\} F_j\left(r'\,|r\right)F_k\left(s'\,|s\right)\right]=1 \\ \text{for some } (j,k)\in D\times D \end{array}\right\}$$

Similarly an action pair $(j,k)\in D\times D$ is said to validate R_h if there exists some $(r,s)\in S\times S$, such that taking the actions moves the pair of states next period into R_g for some $g<h$. Because there are only $S(S-1)/2$ distinct paired states, it follows that $h<S(S+1)/2$ and hence is finite. Notice that validating action pairs might be associated with stochastic transition matrices, but the support of the matrix is a proper subset of $S\times S$. For example if (j,k) validates R_h from (r,s) and $(r',s')\in R_h$, then $F_j\left(r'\,|r\right)+F_k\left(s'\,|s\right)=0$.

A sequential game of incomplete information or a dynamic optimization problem of the type defined in the previous section, exhibits finite dependence if R, the union of the R_h sets, covers A, that is

$$A\subseteq R\equiv \bigcup_{h=1}^{\rho} R_h$$

for some $\rho<S(S+1)/2$. By definition a problem with finite dependence ρ boasts a decision rule $\lambda(s)$, a mapping from S to D, that validates R_h for each $h\in\{1,\ldots,\rho\}$, implying that if the rule is followed, the pair of state returns to the diagonal after ρ periods at most, after an initital one period displacement from the diagonal into A. The rule is not unique since there are no restrictions placed on behavior in the complement of R (with respect to $S\times S$) but this is immaterial for our purposes. We let $F_\lambda\left(s'\,|s\right)$ denote the probability that the state moves from s to s' under $\lambda(s)$.

Estimation of the structrual parameters is based on restrictions that the conditional choice probabilities satisfy under optimization, namely

$$\begin{aligned}\xi_k(s) &= \int d_{kt}(s,\varepsilon_t)\, dG(\varepsilon_t) \\ &= Pr\{v_k(s) - v_l(s) \geq u_l(s) - u_k(s) \text{ for all } l \in \{1,\ldots,K\} \,|s\}\end{aligned}$$

We now show how the inversion theorem of Hotz and Miller can be used in conjunction with finite dependence to form a tractable expression for the differences in the conditional valuation functions, $v_k(s) - v_l(s)$. Given an action pair (k,l) the probability of reaching (s', s'') is $F_k(s'\,|s)\,F_l(s''\,|s)$ while the probability of reaching (s'', s') is $F_k(s'\,|s)\,F_l(s''\,|s)$. Summing over (s', s'') for say $s' < s''$ we have

$$v_k(s) - v_l(s) = \sum_{s''=1}^{S} \sum_{s'=1}^{s"} \left[v(s') - v(s'')\right] \left[F_k\left(s'\,|s\right) F_l\left(s''\,|s\right) - F_k\left(s'\,|s\right) F_l\left(s''\,|s\right)\right]$$

For any decision rule $\lambda(s)$ the value function can be written as

$$\begin{aligned}v(s) &= \sum_{k=1}^{K} \{\xi_k(s)\,(u_k(s) + w_k[\xi(s)] + v_k(s))\} \\ &= \sum_{k=1}^{K} \left\{\xi_k(s)\left(u_k(s) + w_k[\xi(s)] + \psi_{k,\lambda(s)}[\xi(s)]\right)\right\} + v_{\lambda(s)}(s)\end{aligned}$$

The second line uses the definition of $\psi_{k\lambda(s)}[\xi(s)]$ as the the difference $v_k(s) - v_{\lambda(s)}(s)$. In words, the agent must be compensated an amount

$$\sum_{k=1}^{K} \xi_k(s)\,\psi_{k,\lambda(s)}[\xi(s)]$$

in order to induce him to take action $\lambda(s)$ at state s resulting in a continuation value of $v_{\lambda(s)}(s)$. From the definition of $v_{\lambda(s)}(s)$ and $v(s')$ we now obtain

$$\begin{aligned}v(s) &= \sum_{k=1}^{K} \left\{\xi_k(s)\left(u_k(s) + w_k[\xi(s)] + \psi_{k\lambda(s)}[\xi(s)]\right)\right\} + \sum_{r=1}^{S} \beta v(r)\, F_\lambda^{(1)}(r\,|s) \\ &= \sum_{t=1}^{T} \sum_{r=1}^{S} \sum_{k=1}^{K} \beta^t \xi_k(r) \left\{u_k(r) + w_k[\xi(r)] + \psi_{k\lambda(r)}[\xi(r)]\right\} F_\lambda^{(t)}(r\,|s)\end{aligned}$$

The last line telescopes the first expression using an induction, and in the infinite horizon case exploits the fact that $\beta < 1$ and the expected utility terms are bounded.

Suppose the problem has finite dependence of period ρ, validated by choice rule $\lambda(s)$. If we difference the value function evaluated at $(s', s'') \in A$, then by definition the terms beyond ρ drop out and we are left with

$$v(s') - v(s'') = \sum_{t=1}^{\rho} \sum_{r=1}^{S} \sum_{k=1}^{K} \beta^t \left\{\xi_k(r)\,(u_k(r) + w_k[\xi(r)]) + \xi_k(r)\,\psi_{k\lambda(r)}[\xi(r)]\right\} \left[F_\lambda^{(t)}\left(r\,|s'\right) - F_\lambda^{(t)}\left(r\,|s''\right)\right]$$

Successively substituting the expression for $v(s') - v(s'')$ into the expression for $v_k(s) - v_l(s)$ and then the resulting expression for $v_k(s) - v_l(s)$ into the expression fot the condtional choice probabilities, we obtain an expression that is is easy to eavluate if the probability itself can be cheaply computed.

In Altug and Miller (1998) the state variables follow a deterministic law of motion, action $k \in \{1, \ldots, K\}$ inducing the state to move from say s_0 to s_{0k}. Given finite dependence of length ρ with validating rule λ, we denote by $\{\lambda_{1k}, \ldots, \lambda_{\rho k}\}$ the sequence of choices prescribed by λ over the next ρ periods that ensure the state will reach $s_{0\rho}$ on date $t + \rho + 1$ given its initial value of s_{0k}, and denote the intermediate sequence of states associated with $\{\lambda_{1k}, \ldots, \lambda_{\rho k}\}$ by $\left\{s_{0k}^{(1)}, \ldots, s_{0k}^{(\rho)}\right\}$. Consequently the expected valuation function simplifies to

$$v(s_{0k}) = \sum\nolimits_{t=1}^{\rho} \beta^t \left[u_{\lambda_{tk}}\left(s_{0k}^{(t)}\right) + w_{\lambda_{tk}}\left(s_{0k}^{(t)}\right) + \sum\nolimits_{j=1}^{K} \xi_j\left(s_{0k}^{(t)}\right)\psi_{j,\lambda_{tk}}\left[\xi\left(s_{0k}^{(t)}\right)\right]\right] + \beta^{\rho+1} v(s_{0\rho})$$

Differencing $v(s_{0k})$ and $v(s_{0l})$, the $\beta^{\rho+1} v(s_{0\rho})$ term immediately drops out. Since the law of motion for the state variables is deterministic $v_k(s_0) = v(s_{0k})$ implying

$$\xi_k(s) = Pr\left\{\begin{array}{l} \sum\limits_{t=1}^{\rho} \beta^t \left[\begin{array}{l} u_{\lambda_{tk}}\left(s_{0k}^{(t)}\right) - u_{\lambda_{tl}}\left(s_{0l}^{(t)}\right) + w_{\lambda_{tk}}\left(s_{0k}^{(t)}\right) - w_{\lambda_{tl}}\left(s_{0l}^{(t)}\right) \\ + \sum_{j=1}^{K}\left(\xi_j\left(s_{0k}^{(t)}\right)\psi_{j,\lambda_{tk}}\left[\xi\left(s_{0k}^{(t)}\right)\right] - \xi_j\left(s_{0l}^{(t)}\right)\psi_{j,\lambda_{tl}}\left[\xi\left(s_{0l}^{(t)}\right)\right]\right) \end{array}\right] \geq u_l(s) - u_k(s) \\ \hfill \text{for all } l \in \{1, \ldots, K\} \,|s \end{array}\right\}$$

The estimator for the discrete choice part of their framework is based on this expression, and also assumes that the state $s \in S$ is observed. Our econometric contribution develops, for a much larger class of dynamic optimization problems and games with incomplete information, a computationally feasible estimator for frameworks where the states s and their transitions $F_k(s' \,|s)$ are only partially observed.

4 Unobserved Heterogeneity, CCP's, and Finite Dependence

This section explains the algorithm we use for estimating dynamic optimization problems and games of incomplete information where there is unobserved heterogeneity that evolves over time as a stochastic process, and the finite dependence assumption is satisfied. We consider a panel data set of N individuals, T choices for each individual $n \in \{1, \ldots, N\}$ are observed, along with a subvector of their state variables. Observations are independent across individuals. We partition the state variables s_{nt} into those observed by the econometrician $x_{nt} \in \{x_1, \ldots x_X\}$ and, with some abuse of notation, those that are not observed $s_{nt} \in \{s_1, \ldots s_S\}$. The n^{th} individual's unobserved type s_{nt} at time t may affect both the utility function and the transition functions on the observed variables and may also evolve over time. Suppose the initial probability of being assigned to state s_i is π_i. States follow a Markov process with p_{jk} dictating the probability of transitioning from state or type j to type k.[3] The structural parameters that define the utility outcomes for the problem are denoted by $\theta_1 \in \Theta_1$ and the set of conditional choice probabilities (the ξ's) are treated as nuisance parameters in the estimation.

Let $\mathcal{L}\left(d_{nt} \left| x_{nt}, s; \theta_1, \pi, p, \xi\right.\right)$ be the likelihood of observing individual n make choice d_{nt} at time t, conditional on being in state (x_{nt}, s), given structural parameter θ_1 and nuisance parameter ξ. Forming their product over the T periods we obtain the likelihood of any given path of choices and $(d_{n1} \ldots, d_{nT})$, conditional on the $(x_{n1} \ldots, x_{nT})$ sequence and the unobserved state variables $(s(1) \ldots, s(T))$. Integrating the product over the inital unobserved state with probability distribution π and the subsequent probability transitions p then yields the likelihood of observing the choices d_n conditional on x_n given (θ_1, π, p, ξ) :

$$L\left(d_n \left| x_n, \theta_1, \pi, p, \xi\right.\right) \equiv \sum_{s(1)}^{S} \sum_{s(2)}^{S} \ldots \sum_{s(T)}^{S} \pi_{s(1)} \mathcal{L}\left(d_{n1} \left| x_{n1}, s(1); \theta_1, \pi, p, \xi\right.\right) \prod_{t=2}^{T} p_{s(t-1),s(t)} \mathcal{L}\left(d_{nt} \left| x_{nt}, s(t); \theta_1, \pi, p, \xi\right.\right)$$

Therefore the log likelihood for the sample is:[4]

$$\sum_{n=1}^{N} \log L\left(d_n \left| x_n, \theta_1, \pi, p, \xi\right.\right) \tag{1}$$

Directly maximizing the log likelihood above is typically very costly. However, in time series models with regime-switching, Hamilton has (1990) shown that the EM algorithm simplifies this optimization problem substantially. An alternative to maximizing (1) directly is to iteratively maximize the expected log likelihood:

$$\sum\nolimits_{n=1}^{N} \sum\nolimits_{s=1}^{S} \sum\nolimits_{t=1}^{T} q_{nst}^{(m)} \log \mathcal{L}\left(d_{nt} \left| x_{nt}, s; \theta_1, \pi^{(m)}, p^{(m)}, \xi^{(m)}\right.\right) \tag{2}$$

with respect to θ_1 to obtain $\theta_1^{(m+1)}$ until convergence is attained, by treating $q_{1nst}^{(m)}$ as a fixed scalar and $\left(\pi^{(m)}, p^{(m)}, \xi^{(m)}\right)$ as a fixed vector in each optimization step. Here, $q_{1nst} \equiv q_{1st}\left(d_n, x_n, \theta_1, \pi, p, \xi\right)$

[3]Note that this nests both the case where unobserved heterogeneity is permanent (the transition matrix would be an $S \times S$ identity matrix) and the case where the unobserved state is completely transitory (the transition matrix would have the same values in row j as in row k for all j and k).

[4]Note that when unobserved heterogeneity is permanent the log likelihood for the sample is given by:

$$\sum_{n=1}^{N} \log \left(\sum_{s=1}^{S} \prod_{t=1}^{T} \pi_s \mathcal{L}_{nst} \right)$$

is formally defined below as the probability that individual n is in state s at time t given the parameter values $\left(\theta_1^{(m)}, \pi^{(m)}, p^{(m)}, \xi^{(m)}\right)$ conditional on the all the data about n, which we denote by $(d_n, x_n) \equiv (d_{n1}, x_{n1}, \ldots, d_{nT}, x_{nT})$. Finally, $\xi^{(m)}$ is a vector of conditional choice probability estimates plugged into the m^{th} iteration and updated as described below.

Let (θ_1^*, ξ^*) denote the converged values of the structural and nuisance parameters from the EM algorithm. Following the same arguments as given in Arcidiacono and Jones (2003), the EM solution satisfies the first order conditions derived from maximizing (1) with respect to θ_1 given ξ^*. The value of taking the EM approach is that for many paramterizations, each step has a closed form solution, and only low dimensional matrix inversions are required. We now formally define the mapping $q_{nst}\left(d_n, x_n, \theta_1, \pi, p, \xi\right)$, the iterations $\left(\theta_1^{(m)}, \pi^{(m)}, p^{(m)}, \xi^{(m)}\right)$, show how the likelihood is evaluated, and then summarize the steps.

To define $q_{nst}\left(d_n, x_n, \theta_1, \pi, p, \xi\right)$, let $L_{nsr}\left(d_n \left| x_n, \theta_1, \pi, p, \xi\right.\right)$ denote the joint probability of state s occurring at date r for the n^{th} individual and observing the choice sequence d_n, conditional on the exogenous variables x_n, when the structural parameters take value (θ_1, π) and nuisance parameters ξ. Abbreviating $\mathcal{L}\left(d_{nt} \left| x_{nt}, s; \theta_1, \xi\right.\right)$ by $\mathcal{L}_{nts}$, we thus define $L_{rs}\left(d_n \left| x_n, \theta_1, \pi, \xi\right.\right)$ as

$$\begin{aligned} & L_{nrs}\left(d_n \left| x_n, \theta_1, \pi, p, \xi\right.\right) \\ = & \sum_{s(1)}^{S} \cdots \sum_{s(r-1)}^{S} \sum_{s(r+1)}^{S} \cdots \sum_{s(T)}^{S} \pi_{s(1)} \mathcal{L}_{n,1,s(1)} p_{s(r-1),s} \mathcal{L}_{1=nrs} p_{s,s(r+1)} \mathcal{L}_{n,r+1,s(r+1)} \left(\prod\nolimits_{t=2,t\neq r,t\neq r+1}^{T} p_{s(t-1),s(t)} \mathcal{L}_{1n,t,s(t)}\right) \end{aligned}$$

where the summations of $s(1)$ and so on are over $s \in \{s_1, \ldots, s_S\}$. Summing over all states $s \in S$ at time r returns the likelilihood of observing the choices d_n conditional on x_n given (θ_1, π, ξ) :

$$L\left(d_n \left| x_n, \theta, \pi, p, \xi\right.\right) = \sum_{s'=1}^{S} L_{nrs'}\left(d_n \left| x_n, \theta_1, \pi, p, \xi\right.\right)$$

Therefore the probability that individual n is in state s at time r given the parameter values (θ_1, π, p, ξ) conditional on all the data for n is:

$$q_{nrs}\left(d_n, x_n, \theta_1, \pi, p, \xi\right) \equiv \frac{L_{nrs}\left(d_n \left| x_n, \theta_1, \pi, p, \xi\right.\right)}{L_n\left(d_n \left| x_n, \theta_1, \pi, p, \xi\right.\right)} \tag{3}$$

To make the algorithm operational we must explain how to update the probabilities for the unobserved states $((\pi, p))$, the other structural parameters (θ_1), and the conditional choice probabilities (ξ). The updating formula for the transition probabilities is based on the identities:

$$p_{jk} \equiv \Pr\left\{k \left| j\right.\right\} = \frac{\Pr\{k, j\}}{\Pr\{j\}} = \frac{E\left\{E\left[s_{ntk} \left| d_n, x_n, s_{n,t-1,j}\right.\right] E\left[s_{n,t-1,j} \left| d_n, x_n\right.\right]\right\}}{E\left\{E\left[s_{ntj} \left| d_n, x_n\right.\right]\right\}} \equiv \frac{E\left[q_{ntk|j} q_{ntj}\right]}{E\left[q_{ntj}\right]}$$

where $q_{nts|j} \equiv E\left[s_{ntk}\left|d_n, x_n, s_{n,t-1,j}\right.\right]$ denotes the probability of individual n being type k at time t conditional on the data and also on being type j at time $t-1$ and is defined by the expression:

$$q_{ntk|j} = \frac{p_{jk}\mathcal{L}_{ntk}\left(\sum_{s(t+1)}^{S} \cdots \sum_{s(T)}^{S} \prod_{t'=t+1}^{T} p_{s(t'-1),s(t')}\mathcal{L}_{n,t,s(t')}\right)}{\sum_{s'}^{S} p_{js'}\mathcal{L}_{nts'}\left(\sum_{s(t+1)}^{S} \cdots \sum_{s(T)}^{S} \prod_{t'=t+1}^{T} p_{s(t'-1),s(t')}\mathcal{L}_{n,t,s(t')}\right)}$$

Averaging over $q_{nkt|j}q_{njt}$ to approximate the joint probability $E\left[q_{ntk|j}q_{ntj}\right]$, and averaging over q_{njt} to estimate $E\left[q_{ntj}\right]$, we update p with the formula

$$p_{jk}^{(m+1)} = \frac{\sum_{n=1}^{N}\sum_{t=2}^{T} q_{ntk|j}^{(m)} q_{ntj}^{(m)}}{\sum_{n=1}^{N}\sum_{t=2}^{T} q_{ntj}^{(m)}} \tag{4}$$

Setting $r = 1$ yields the conditional probability of the n^{th} individual being in unobserved state s in the first time period. We update π by averaging those conditional probabilities obtained from the previous iteration over the sample population:

$$\pi_s^{(m+1)} = \frac{1}{N}\sum_{n=1}^{N} q_{n1s}\left(d_n, x_n, \theta_1^{(m)}, \pi^{(m)}, p^{(m)}, \xi^{(m)}\right) \tag{5}$$

In a Markov stationary environment, the unconditional probabilities reproduce themselves each period. In that special case we can average over all the periods in the sample in the update formula for π to obtain

$$\pi_s^{(m+1)} = \frac{1}{NT}\sum_{t=1}^{T}\sum_{n=1}^{N} q_{nst}\left(d_n, x_n, \theta_1^{(m)}, \pi^{(m)}, p^{(m)}, \xi^{(m)}\right)$$

The last component in the updating formula is the vector of conditional choice probabilities. In contrast to models where there is no unobserved heterogeneity, initial consistent estimates of ξ cannot be cheaply computed prior to structural estimation, but must be iteratively updated along with (θ_1, π, p). One way of updating the nuisance parameters is to substitute in the likelihood evaluated at the previous iteration. Let $\mathcal{L}_k(x, s; \theta_1, \xi)$ denote the conditional likelihood of observing choice $k \in \{1, \ldots, K\}$ for the state (x, s) when the parameters are (θ_1, ξ). An updating rule for ξ is:

$$\xi_{kxs}^{(m+1)} = \mathcal{L}_k\left(x, s; \theta_1^{(m+1)}, \xi^{(m)}\right) \tag{6}$$

Another way of updating ξ comes from exploiting the identities

$$\Pr\left\{d_{ntk}\left|x, s\right.\right\}\Pr\left\{s\left|x\right.\right\} = \Pr\left\{d_{ntk}, s\left|x\right.\right\} \equiv E\left[d_{ntk}s\left|x\right.\right] = E\left[d_{ntk}E\left\{s\left|d_n\right.\right\}\left|x\right.\right]$$

where the last equality follows from law of iterated expections, plus the definition of d_n, which includes d_{ntk} as a component. From its definition $q_{1nts} = E\left[s\left|d_n\right.\right]$, and since $\Pr\left\{s\left|x\right.\right\} = E\left\{E\left[s\left|d_n\right.\right]\left|x\right.\right\}$ by the law of iterated expectations, it now follows that

$$\Pr\left\{d_{ntk}\left|x, s\right.\right\} = \frac{E\left[d_{ntk}q_{nts}\left|x\right.\right]}{E\left[q_{nts}\left|x\right.\right]}$$

In words, of the fraction of the total population with characteristic x in state s, the portion choosing the k^{th} action is ξ_{kxs}. This identity suggests a second way of updating ξ, namely:

$$\xi_{kxs}^{(m+1)} = \frac{\sum_{n=1}^{N} d_{nkt} q_{1nts}^{(m)} I(x = x_{nt})}{\sum_{n=1}^{N} q_{1nts}^{(m)} I(x = x_{nt})} \tag{7}$$

where $I(x = x_{nt})$ where I is the indicator function for x. This method is particularly useful when an outcome from one of the choices is observed, so that the estimation of the (π, p) probabilities, which determine the stochastic process for unobserved heterogeity, can be separated from the estimation of the structural parameters that define the utility obtained for the remaining choices, a variation on our basic framework that we develop below.

We have now defined all the pieces necessary to implement the algorithm. It is triggered by setting initial values for the conditional choice probabilities, $\xi^{(1)}$, the initial distribution of the the unobserved states, $\pi^{(1)}$, their probability transitions, $p^{(1)}$, initial values for the structural parameters, $\theta_1^{(1)}$, and the conditional choice probabilities $\xi^{(1)}$.The values for $\theta_1^{(1)}$ and $\xi^{(1)}$ can be obtained from estimating a model without any unobserved heterogeneity and perturbing the estimates obtained. Each iteration in the algorithm has four steps. Given $\left(\theta_1^{(m)}, \pi^{(m)}, p^{(m)}, \xi^{(m)}\right)$ the $(m+1)^{th}$ proceeding as follows:

Step 1 Compute $q_{nst}^{(m+1)}$, the probability that each individual n is of type s at each time period t for all S types conditional upon (d_n, x_n), using (3) with parameters $\left(\theta_1^{(m)}, \pi^{(m)}, p^{(m)}, \xi^{(m)}\right)$. Also compute $q_{1nts|j}^{(m+1)}$, the probability that each individual n is of type s at each time period t for all S types conditional on being type j in period $t-1$.

Step 2 Given $q_{nts}^{(m+1)}$ and $q_{nts|j}^{(m+1)}$for each (n, t, s, j) compute $\pi^{(m+1)}$ and $p^{(m+1)}$ using (5) and (4).

Step 3 Given $q_{nts}^{(m+1)}$ for each (n, t, s, j), and $\xi^{(m)}$, maximize the expected log likelihood function to obtain estimates of $\theta_1^{(m+1)}$.

Step 4 Update the conditional choice probability nuisance parameters $\xi^{(m+1)}$, using either (6) evaluated at $\left(\theta_1^{(m+1)}, \xi^{(m)}\right)$, or (7) evaluated at $q_{nts}^{(m+1)}$ for each (n, t, s).

While this is only one of several possible orderings of steps within an iteration that converges to the same fixed point, this procedure has the virtue of only using the $q_{1nst}^{(m)}$ values within each iteration, as opposed to using both sets of $q_{1nst}^{(m)}$ and $q_{1nst}^{(m+1)}$ within an iteration.

5 Unobserved Heterogeneity, CCP's, and Outcomes

Although the estimation method described above is computationally feasible for many problems where finite time dependence holds, not all dynamic discrete choice model have that property. We now show how to estimate the distribution of unobserved heterogeneity without fully specifying the dynamic optimization problem provided that there is data on an outcome that depends upon the unobserved heterogeneity. For example, when the conditional transition probability for the observed state variables depends on the current values of the unobserved state variables, when there is data on a payoff of a choice that depends on the unobserved heterogeneity, or when data exists on some other outcome that is determined by the unobserved state variables, the estimation approach above can be extended beyond the finite dependence case by using the two stage procedure developed here. One advantage of using this two stage procedure is that it enlarges the class of models to which our estimator applies. Rather than assuming the model exhibits finite time dependence, one could estimate a stationary markov model without this property by using the estimator described here to estimate the distribution of unobserved heterogeneity in a first stage. The estimates from the first stage can then be combined with non-likelihood based estimation methods in the second stage.

This adaptation is motivated by the second method of updating the conditional choice probabilities, using equation (7), which can be interpreted as an empirical estimator of the fraction of people in any given state making a particular choice. When information is available on both the individual choices and an outcome, this method for updating the conditional choice probabilities implies that we can substitute the empirical estimator into the likelihood for observing a sequence of outcomes without estimating all the structural parameters that affect the decision itself.

More formally, decompose the likelihood into two parts: the likelihood associated with a particular outcome conditional on the choice, $\mathcal{L}_1$, and the likelihood associated with the choice, $\mathcal{L}_2$. Let the parameters in each of the likelihoods be given by θ_1 and θ_2 where θ_1 and θ_2 may overlap. We then replace $\mathcal{L}_{nst}$ in (3) with $\mathcal{L}_{1nst}\mathcal{L}_{2nst}$. But now instead of using the likelihoods themselves for the $\mathcal{L}_{2nst}$'s, substitute in the weighted CCP's along the lines of (7) by using:

$$\hat{\mathcal{L}}_{2nst} = \frac{\sum_{n=1}^{N} q_{nst} d_{nkt} I(x = x_{nt})}{\sum_{n=1}^{N} q_{nst} I(x = x_{nt})} \tag{8}$$

By iterating on (3) and (8) until a fixed point is reached, we obtain estimates of the conditional probability of being in a particular state given 1) current values of the π's and p_{jk}'s, 2) the current values of θ_1, and 3) the weighted average of the data governing the choices where the weights are given by the conditional probabilities of being in particular states. The first stage of the algorithm iterates on three steps where the $m+1$ iteration follows:

Step 1 For each of the S types, calculate the conditional probability that each individual is of type s at each time period t, $q_{nst}^{(m+1)}$ using (3) and substituting in for $\mathcal{L}_2$ with (8). Iterate on (3) and (8) until convergence.

Step 2 Given the $q_{nst}^{(m+1)}$'s, the $\pi_i^{(m)}$'s, and the $p_{ij}^{(m)}$'s, obtain the $\pi_i^{(m+1)}$'s and the $p_{ij}^{(m+1)}$ using (5) and (4).

Step 3 Maximize the expected log likelihood function to obtain estimates of $\theta_1^{(m+1)}$ and taking the $q_{nst}^{(m+1)}$'s as given.

Once convergence has been achieved, we have consistent estimates of the conditional probability of being in state s at time t given the data. As we prove below, subject to the outcome model being identified, the first stage of estimation yields consistent estimates of the probability distributions characterizing unobserved heterogeneity (π, p) along with the estimates of the conditional choice probability incidental parameters ξ. Having achieved convergence in the first stage, there are several methods for estimating θ_1, the preference parameters determining the (remaining) preferences over choices by substituting $\left(\widehat{\pi}, \widehat{p}, \widehat{\xi}\right)$ into parameters in a second stage. One approach we now briefly describe appeals directly to the representation theorem, which yields the $K-1$ equalities

$$\psi\left[\xi\left(x\right)\right] = \begin{bmatrix} \psi_2\left[\xi\left(z\right)\right] \\ \vdots \\ \psi_K\left[\xi\left(z\right)\right] \end{bmatrix} = \begin{bmatrix} v_2\left(z\right) - v_1\left(z\right) \\ \vdots \\ v_K\left(z\right) - v_1\left(z\right) \end{bmatrix}$$

$$\psi\left[\xi_{xs}\right] = \begin{bmatrix} \psi_2\left[\xi\left(z\right)\right] \\ \vdots \\ \psi_K\left[\xi\left(z\right)\right] \end{bmatrix} = \begin{bmatrix} v_2\left(x, s; \theta, \pi, p\right) - v_1\left(z\right) \\ \vdots \\ v_K\left(z\right) - v_1\left(z\right) \end{bmatrix}$$

for each $z \equiv (x, s)$. If the model satisfies finite dependence, then the appropriate representation can be used for $v_k(z)$ to express the conditional valuation functions, using standard nonlinear optimziation methods. Alternatively, the simulation estimators of Hotz, Miller, Sanders and Smith or Bajari, Bankard and Levin can be directly applied regardless of whether the model satisifes the limited dependence property or not, providing the outocme equation is not subject to selection considerations. To implement the former, we sequentially simulate, for each choice $k \in \{1, \ldots, K\}$ and each state $z_0 \in \{z_1, \ldots, z_K\}$ say $I \geq 1$ future paths of the state variables $\left(z_1^{(i,k,z_0)}, z_2^{(i,k,z_0)}, \ldots\right)$and choices $\left(d_1^{(i,k,z_0)}, d_2^{(i,k,z_0)}, \ldots\right)$ using $\widehat{p}$, the conditional transition given current choices and next period's observed state variables, and F, the marginal transition for the observed variables, as well as the estimated conditional choice probabilities, $\widehat{\xi}$. Thus $z_1^{(i,k,z_0)}$ is a random draw from $\widehat{p}_{z_0} \equiv (\widehat{p}_{z_0 s_1}, \ldots, \widehat{p}_{z_0 s_S})$, while $d_1^{(i,k,z_0)}$ is randomly drawn from the multinomial distribution the $\widehat{\xi}_{z_0} \equiv \left(\widehat{\xi}^*_{z_0 s_1}, \ldots, \widehat{\xi}^*_{z_0 s_S}\right)$, and so on. Then we form a future utility path

$$v_{kz_0} = \frac{1}{N}\sum\nolimits_{t=1}^{T}\sum\nolimits_{k=1}^{K}\beta^t p\left(z_t\right) d^{(i)}_{x(t)s(t)}\left[u_k\left(x_t, s_t; \theta_1, \widehat{\theta}_2\right) + w_k\left(\widehat{\xi}\right)\right]$$

To generate the estimates for θ_1 we minimize a quadratic in differences between v_{kz_0} and $q\left[\xi_{xs}\right]$, or equivalently premultiply the vector of differences by a weighting matrix of dimension This generates , and simulating utility paths generated by (π, p, ξ) ,generates estimates that are $\sqrt{N}$ consistent and asymptotically normal.

6 Large Sample Properties

The defining equations for this estimator come from three sources. First are orthogonality conditions for θ, the parameters defining utility and the probability transition matrix for the observed states, which are analogous to the score for a discrete choice random utility model with nuisance parameters are used in defining the payoffs. Second are the orthogonality conditions for the initial distribution of the unobserved heterogeneity π, and its transition probability matrix p, again computed from the likelihood as in a random effects model. Third are the equations which define the nuisance parameters as estimators of the conditional choice probabilities. This section defines the equations that are satisfied by our estimator, proves it is consistent, and establishes its asymptotic distribution.

Let (φ^*, ξ^*) solve the EM algorithm, where $\varphi \equiv (\theta, \pi, p)$ is the vector of structural parameters. For any fixed set of nuisance parameters ξ, the solution to the EM algorithm (equation 9) satisfies the first order conditions of the original problem (1). Consequently setting $\xi = \xi^*$ in the original problem implies the first order conditions for the original problem are satisfied. It now follows that the large sample properties of our estimator can be derived by analyzing the score associated with (1) augmented by a set of equations that solve the conditional choice probability nuisance parameter vector ξ, either the likelihoods or the empirical likelhoods, as discussed in the previous section.

In the previous section we defined the conditional likelihood of (φ, ξ) upon observing d_n given x_n, which we now denote as $\mathcal{L}\left(d_n \left| x_n, s; \varphi, \xi\right.\right) \equiv \mathcal{L}\left(d_n \left| x_n, s; \theta, \pi, p, \xi\right.\right)$. The arguments above imply that in this case (φ^*, ξ^*) solve

$$\frac{1}{N}\sum_{n=1}^{N}\frac{\partial \log\left[L\left(d_n \left| x_n; \varphi^*, \xi^*\right.\right)\right]}{\partial \varphi} = 0$$

When the choice specific likelihood is used to update the nuisance parameters, the definition of the algorithm implies that upon convergence, $\xi^*_{kxs} = \mathcal{L}_k(x, s; \varphi^*, \xi^*)$ for each (k, x, s). Stacking $\mathcal{L}_k(x, s; \varphi^*, \xi^*)$ for each choice k and each value (x, s) of state variables to form $\mathcal{L}(\varphi, \xi)$, a $K \times X \times S$ vector function of the parameters (φ, ξ), our estimator satisfies the KXS additional parametric restrictions $\mathcal{L}(\varphi^*, \xi^*) = \xi^*$. When the empirical likelihoods are used instead, this condition is replaced by the KSX equalities

$$\xi^*_{kxs} \sum_{t=1}^{T} \sum_{n=1}^{N} I(x = x_{nt}) q_{st}(d_n, x_n, \varphi^*, \xi^*) = \sum_{t=1}^{T} \sum_{n=1}^{N} d_{nkt} I(x = x_{nt}) q_{st}(d_n, x_n, \varphi^*, \xi^*)$$

Forming the SX dimensional vector $q_t(d_n, \theta, \varphi, \xi)$ from stacking the terms $I(x = x_{nt}) q_{st}(d_n, x_n, \varphi^*, \xi^*)$ for each state (x, s) and the KSX dimensional vector $q^{(n,t)}_{st}(d_n, \varphi, \xi)$ from $I(x = x_{nt}) q_{st}(d_n, x_n, \varphi, \xi)$, we rewrite this alternative set of restrictions in vector form as

$$\left[\frac{1}{NT} \sum_{t=1}^{T} \sum_{n=1}^{N} q_t(d_n, \varphi^*, \xi^*)\right] H \xi^* = \frac{1}{NT} \sum_{t=1}^{T} \sum_{n=1}^{N} q^{(n,t)}_{st}(d_n, \varphi^*, \xi^*)$$

where H is the $SX \times KSX$ block diagaonal matrix

$$H \equiv \begin{bmatrix} 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \end{bmatrix}$$

We now turn to the properties of the algorithm. First we show that if the model generating the data, φ_0, is identified within the Ψ space, then φ^* is a consistent estimator of φ_0. It follows that if more than one root solves the orthogonality conditions that define the limit points of our estimator, then the model is not identified, so cannot be estimated using maximum likelihood either. In this way the limit properties of the algorithm are informative about whether the underlying framwork is identified. The following proposition formally states this result.

Proposition 1 *Suppose the data $\{d_n, x_n\}$ are generated by φ_0, exhibiting conditional choice probabilites ξ_0. If φ_1 satisfies the vector of moment conditions*

$$E\left[\frac{\partial \log [\mathcal{L}(d_n | x_n; \varphi_1, \xi_1)]}{\partial \varphi}\right] = 0$$

where the expectation is taken over (d_n, x_n) in the sample population and $\mathcal{L}(\varphi_1, \xi_1) = \xi_1$, then under standard regularity conditions φ_0 and φ_1 are observationally equivalent.

Proof The definition of this estimator ensures (π_1, p_1, ξ_1) define a set of probability distributions that generate the empirical distribution of the choices observed in the data conditional on the observed state variables. Thus the data generation process is fully characterized by (φ_1, ξ_1). It only remains to show that in the structural model defined by φ_1, optimizing behavior by the agents leads to the vector of conditional choice probabilities ξ_1.

From the parameter vector (φ_1, ξ_1) we form the payoff function, or expected value function, $v(z; \varphi, \xi_1)$, using the the finite dependence representation discussed in Section 2 and exploited in the estimation of (φ_1, ξ_1). Also let $u_k(z) \equiv u_k(z; \theta)$ denote the component of current utility that depends on the state variables when the k^{th} choice is made and given a parameterziation within the Ψ class of models It follows that, conditional on $(z_{nt}, \varepsilon_{nt})$ the value of choosing k is

$$u_k(z_{nt}; \varphi) + \sum_{z'=1}^{Z} v(z'; \varphi, \xi_1)\, dF_k(z' | z_{nt}; \varphi) + \varepsilon_{nt}$$

We consider the problem of estimating $\varphi \in \Psi$ in a random utility model with payoffs defined above, where ξ_1 is known and ε is parameterized by $G(\varepsilon; \varphi)$, which is known up to φ.

By construction, the likelihood of φ given the sequence (d_n, x_n) and the incidental parameters ξ_1 is $\mathcal{L}(d_n | x_n; \varphi, \xi_1)$. Define φ_1 as a value that solves the expectation of the score in the proposition. Under standard regularity conditions, φ_{ML}, the maximum likelihood estimator of $\varphi \in \Psi$, is defined by its first order condition. Appealing to continuity of the likelihood in φ and the law of large numbers, φ_{ML} converges to φ_1. Furthermore φ_{ML} is consistent, which implies from the definition of ξ_1, that φ_1 solves

$$\xi_1^{(kz)} = \Pr\left\{ \begin{array}{r} \varepsilon_k - \varepsilon_l \geq u_l(z; \varphi) - u_k(z; \varphi) + \sum_{z'=1}^{Z} v(z'; \varphi)\, dF_l(z' | z; \varphi) - \sum_{z'=1}^{Z} v(z'; \varphi, \xi_1)\, dF_k(z' | z; \varphi) \\ \text{for all } l \in \{1, \ldots, K\} \end{array} \right\}$$

for all (k, z). Consequently the conditional choice probabilites ξ_1 are generated by a random utility function above with the parameterization defined above.

From our representation result in Section 2, $v(z; \varphi_1, \xi_1)$ is the expected valuation function for the dynamic programming problem, and therefore the conditional choice probabilites arise from individuals optimizing in the φ_1 model. We conclude that the choice probabilities conditional on the observed state variables for both φ_0 and φ_1 are the same, thus proving φ_0 and φ_1 are observationally equivalent.$\|$

Given the identification of $\varphi_0 \in \Psi$ and hence the consistency of φ^*, its convergence at rate $\sqrt{N}$ can be readily established by appealing well known results in the literature found in Hansen (1982) or Newey and McFadden (1994) for example. In general φ^* does not, however, achieve the Cramer Rao lower bound. To define its asymptotic covariance matrix let

$$h_1(d_n, x_n, \varphi, \xi) \equiv \begin{bmatrix} \partial \log [L(d_n | x_n; \varphi, \xi)] / \partial \varphi \\ \mathcal{L}(\varphi, \xi) - \xi \end{bmatrix}$$

and

$$h_2(d_n, x_n, \varphi, \xi) \equiv \begin{bmatrix} \partial \log [L(d_n | x_n; \varphi, \xi)] / \partial \varphi \\ \left[q(d_n, x_n, \varphi, \xi) H\xi - q^{(n,t)}(d_n, x_n, \varphi, \xi) \right] \end{bmatrix}$$

Respectively defining the expected outer product and derivative matrices for each estimator $k \in \{1, 2\}$ as

$$\begin{aligned} \Omega_k &= E\left[h_k(d_n, x_n, \varphi_0, \xi_0) h_k(d_n, x_n, \varphi_0, \xi_0)' \right] \\ \Gamma_k &= E\left[\frac{\partial h_k(d_n, x_n, \varphi_0, \xi_0)}{\partial \varphi} \frac{\partial h_k(d_n, x_n, \varphi_0, \xi_0)}{\partial \xi} \right] \end{aligned}$$

it follows from Hansen (1982, Theorem 3.1) for example, that $\sqrt{N}(\varphi^* - \varphi_0)$ is asymptotically normally distributed with mean zero and covariance matrix

$$\Sigma_k = [I_\rho : 0_{KSX}] \Gamma_k^{-1} \Omega_k \Gamma_k^{-1'} [I_\rho : 0_{KSX}]'$$

where I_ρ is the ρ dimensional identity matrix, ρ is the dimension of φ, and 0_{KSX} is a $\rho \times KSX$ matrix of zeros.

7 Small Sample Performance

We perform two Monte Carlo simulations to assess the performance of our estimator for finite samples. Both simulations are simpler versions of proposed empirical projects discussed below. The first simulation is of a finite horizon model and shows that both methods of updating the CCP's yield parameter estimates similar to full backwards recursion with little loss in efficiency. The second simulation is an infinite horizon game and shows that CCP estimators can handle regime switching in dynamic discrete choice models even when few time periods are observed in the data.

7.1 Finite Horizon Monte Carlo

The first simulation is a model of teenage behavior where in each period t the teen decides among three alternatives: drop out ($d_{0t} = 1$), stay in school and do drugs ($d_{1t} = 1$), or stay in school but do not do drugs ($d_{2t} = 1$). In this model there are two types of agents, where an individual's type affects their utility of using drugs. The probability of being type L is given by π. An individual learns their type through experimentation: if at any point an individual has tried drugs their type is immediately revealed to them. Hence, the individual's information about their type, s_t, comes from the set $\{U, L, H\}$ where U indicates that the individual does not know their type. The econometrician and the individual then have the same priors as to the individual being a particular type before the individual has tried drugs. However, once an individual has tried drugs their type is immediately revealed to them but not to the econometrician.

The individual also faces a withdrawal cost if he uses drugs at time $t-1$ but does not use at time t. Hence, the other relevant state variable, x_t, is taken form $\{N, Y\}$ which indicate whether the individual used drugs in the previous period. There are then five possible information sets $\{s_t, x_t\}$: $\{U, N\}$, $\{L, N\}$, $\{L, Y\}$, $\{H, N\}$, and $\{H, Y\}$. With dropping out being a terminal state, the evolution of the state space conditional on the two stay in school options is given by the transition matrix in Table 1.

Table 1: Evolution of the State Space†

		$\{U,N\}$	$\{L,N\}$	$\{L,Y\}$	$\{H,N\}$	$\{H,Y\}$
$\{D,U\}$	$d_{2t}=1$	1	0	0	0	0
	$d_{1t}=1$	0	0	π	0	$1-\pi$
$\{L,N\}$	$d_{2t}=1$	0	1	0	0	0
	$d_{1t}=1$	0	0	1	0	0
$\{L,Y\}$	$d_{2t}=1$	0	1	0	0	0
	$d_{1t}=1$	0	0	1	0	0
$\{H,N\}$	$d_{2t}=1$	0	0	0	1	0
	$d_{1t}=1$	0	0	0	0	1
$\{H,Y\}$	$d_{2t}=1$	0	0	0	1	0
	$d_{1t}=1$	0	0	0	0	1

†Rows give state at time t, columns the state at time $t+1$.

With dropping out being a terminal state, we normalize utility with respect to this option. The flow utilities net of the ϵ's when an individual know their type are given by:

$$u_1(s_t \in \{L, H\}, x_t) = \alpha_0 + \alpha_1 + \alpha_2(s_t = H)$$

$$u_2(s_t \in \{L, H\}, x_t) = \alpha_0 + \alpha_3(x_t = Y)$$

where α_0 is the baseline utility of attending school, α_1 is the baseline utility of using drugs, α_2 is the additional utility from being type H and using drugs, and α_3 is the withdrawal cost. Note that if the individual chooses to use drugs no withdrawal cost is paid meaning that x_t is not relevant. On the other hand, if the individual choose not to use drugs only the individual's type becomes irrelevant as type only affects the utility of using drugs. We can then write down the similar expressions for those who do not know their type;

$$u_1(U, x_t) = \alpha_0 + \alpha_1 + (1 - \pi)\alpha_2(s_t = H)$$
$$u_2(U, x_t) = \alpha_0$$

where now the utility of using drugs is probabilistic and, since the individual has not used drugs in the past, no withdrawal cost needs to be paid.

With the flow utilities in hand, we now focus on our attention on the value functions themselves. Here it is important to note that choosing to drop out leads to a terminal state. Combining this with having the ϵ's be distributed Type I extreme value means that the future value components are functions of the transition on the state space and the one period ahead conditional choice probabilities of dropping out. The expressions for the v_k's when an individual knows their type then follow:

$$v_1(s_t \in \{L, H\}, x_t) = u_1(s_t \in \{L, H\}, x_t) - \beta \ln [\xi_0(s_t \in \{L, H\}, Y)] + \beta\gamma$$
$$v_2(s_t \in \{L, H\}, x_t) = u_2(s_t \in \{L, H\}, x_t) - \beta \ln [\xi_0(s_t \in \{L, H\}, N)] + \beta\gamma$$

with the corresponding expressions when an individual does not know their type following:

$$\begin{aligned} v_1(U, x_t) &= u_1(U, x_t) + \beta\gamma \\ &\quad -\beta \left(\pi \ln [\xi_0(L, Y)] + (1 - \pi) \ln [\xi_0(H, Y)]\right) \\ v_2(U, x_t) &= u_2(U, x_t) - \beta \ln [\xi_0(U, N)] + \beta\gamma \end{aligned}$$

For each simulation we create 5000 simulated individuals with 5 periods of data. There are less observations for those who drop out as no further decisions occur once the simulated individual leaves school. We estimate the model using three different methods of calculating the expected future utility where all three are equivalent asymptotically. The first calculates the expected future utility via backwards recursion while the second and third use CCP's with the CCP's updated using the likelihoods or the weighted data respectively. Each simulation was performed 500 times. Table 2 shows that both CCP estimators performed nearly as well as the more efficient model. Updating the CCP's via the likelihoods yielded smaller standard errors than using the data but the differences were small. Note that it is particularly surprising how well the CCP estimators performed given that we are only using data from a discrete outcome. In more standard cases unobserved heterogeneity will affect both transitions (which may be on continuous variables) and choices. Using the variation from the transitions is likely to further reduce the differences across the estimators.

Table 2: School and Drug Choice Monte Carlo†

	School/No Drug Intercept	No Withdrawal Benefit	Low Drug Type	High Drug Type	Discount Factor	Prob. of High Type
True Parameters	0.2	0.4	-15	2	0.9	0.7
Efficient Estimates	0.197	0.403	-15.011	2.003	0.900	0.700
Standard Error	0.069	0.078	0.755	0.074	0.017	0.014
CCP Estimates 1‡	0.194	0.397	-14.835	1.99	0.891	0.701
Standard Error	0.085	0.092	0.800	0.087	0.017	0.014
CCP Estimates 2	0.2003	0.3934	-14.833	2.000	0.884	0.7005
Standard Error	0.097	0.105	0.817	0.099	0.026	0.014

†Listed values are the means and standard errors of the parameter estimates over the 500 simulations.

‡CCP Estimates 1 refers to updating the CCP's via the likelihoods while CCP Estimates 2 updates the CCP's using the data directly.

7.2 Monte Carlo 2: An Infinite Horizon Entry/Exit Game

Our second Monte Carlo examines an entry/exit game along the lines of the game described in section 4.3. We assume that the econometrician observes prices as well, though these prices do not affect the firm's expected profits once we control for the relevant state variables. We specify the price equation as:

$$Y_{jt} = \alpha_0 + \alpha_1(1 \text{ Firm in Market } j) + \alpha_2(2 \text{ Firms in Market } j) + \alpha_3 s_{jt} + \zeta_{jt} \tag{9}$$

where j indexes the market. Price then depends upon how many firms are in the market, an unobserved state variable that transitions over time, s_{jt}, and takes on one of two values, H or L, as well as a normally distributed error, ζ_{jt}. Firms know the current value of this unobserved state variable but only have expectations regarding its transitions. The econometrician has the same information as the firms regarding the probability of transitioning from state to state but does not know the current value of the state. Profits are assumed to be linear in whether the firm has a competitor and the state of the market. Each of our Monte Carlos has 3000 markets[5] observed for 5 periods each. The rest of the specification of the Monte Carlo follows directly from the entry/exit game discussed in section 4.3.

The first set of columns corresponds to our base CCP algorithm described in section 4. The second set involves estimating the parameters governing the unobserved variables, the conditional probabilities of being in each of the unobserved states, and the parameters of the price equation in a first stage. The likelihood contributions of the entry/exit choices are calculated from the data in this first stage as opposed to solving out the dynamic discrete choice problem. We then use the conditional probabilities of being in a particular state as weights in a second stage maximization of the dynamic discrete choice problem.

Results are presented for 100 simulations in Table 2. Both the base CCP estimator and the CCP estimator where unobserved heterogeneity is estimated in a first stage yield estimates that are quite close to the truth with small standard errors. The noisiest parameters are those associated with the persistence of the states and with the initial conditions. Of particular interest are the coefficients on monopoly and duopoly in the demand equation. If we ignore unobserved heterogeneity and estimate the demand by OLS, the coefficients are -.18 and -.41 respectively, biased significantly upward compared to the true values of -.3 and -.7. Controlling for dynamic selection shows a much stronger effect of adding firms to the market, consistent with the actual data generating process.

8 Conclusion

Estimation of dynamic discrete choice models is computationally costly, particularly when controls for unobserved heterogeneity are implemented. CCP estimation provides a computationally cheap way of estimating dynamic discrete choice problems. In this paper we have broadened the class of CCP estimators that rely on a small number of CCP's for estimation and have shown how to incorporate unobserved heterogeneity that transitions over time. The algorithm itself borrowed

[5] This is roughly the number of U.S. counties.

Table 3: Dynamic Entry/Exit Simulations†

			Method 1§		Method 2	
		True Values	Estimates	Std. Error	Estimates	Std. Error
	Intercept L	7.000	6.999	0.042	7.007	0.051
Price	Intercept H	8.000	8.005	0.066	8.005	0.069
Equation	1 Firm	-0.300	-0.299	0.035	-0.306	0.044
	2 Firms	-0.700	-0.702	0.053	-0.702	0.059
	Flow Profit L	0.000	-0.002	0.032	-0.003	0.044
Profit	Flow Profit H	0.500	0.516	0.103	0.501	0.080
Function	Duopoly Cost	-1.000	-1.009	0.043	-0.994	0.053
	Entry Cost	-1.500	-1.495	0.027	-1.506	0.030
	$p_{LL}^{\ddagger}$	0.800	0.799	0.028	0.804	0.029
Unobserved	p_{HH}	0.700	0.702	0.040	0.702	0.028
Heterogeneity	$\pi_L^{\dagger\dagger}$	0.800	0.799	0.031	0.802	0.045

† 100 simulations of 3000 markets for 5 periods. β set at 0.9.

‡ The probability of a market being in the low state in period t conditional on being in the low state at $t-1$.

†† Initial probability of a market being assigned the low state.

§ Method 1 uses our base algorithm. Method 2 estimates unobserved heterogeneity using two-stage method taking the distribution of unobserved heterogeneity as given when solving the dynamic discrete choice problem.

from the macroeconomics literature on regime switching– and in particular the insights gained from the EM algorithm– in order to form an estimator that iterated on 1) updating the conditional probabilities of being a particular type, 2) updating the CCP's, 3) forming the expected future utility as functions of the CCP's, and 4) maximizing a likelihood function where the expected future utility is taken as given. The algorithm was shown to be both computationally simple with little loss of information relative to full information maximum likelihood.

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